

Information geometry and the hydrodynamical formulation of quantum mechanics

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Abstract

Let (M, g) be a compact, connected and oriented Riemannian manifold with volume form $d\text{vol}_g$. We denote \mathcal{D} the space of smooth probability density functions on M , i.e. $\mathcal{D} := \{\rho \in C^\infty(M, \mathbb{R}) \mid \rho > 0 \text{ and } \int_M \rho \cdot d\text{vol}_g = 1\}$.

In this paper, we show that the Fréchet manifold \mathcal{D} is equipped with a Riemannian metric $g^\mathcal{D}$ and an affine connection $\nabla^\mathcal{D}$ which are infinite dimensional analogues of the Fisher metric and exponential connection in the context of information geometry. More precisely, we use Dombrowski's construction together with the couple $(g^\mathcal{D}, \nabla^\mathcal{D})$ to get a (non-integrable) almost Hermitian structure on $T\mathcal{D}$, and we show that the corresponding fundamental 2-form is a symplectic form from which it is possible to recover the usual Schrödinger equation for a quantum particle living in M .

These results echo a recent paper of the author where it is stressed that the Fisher metric and exponential connection are related (via Dombrowski's construction) to Kähler geometry and quantum mechanics in finite dimension.

Introduction

A *statistical manifold* defined over a measured space (Ω, dx) is a manifold S together with an injection

$$j : S \hookrightarrow \left\{ p : \Omega \rightarrow \mathbb{R} \mid p \text{ is measurable, } p \geq 0 \text{ and } \int_\Omega p(x) dx = 1 \right\}. \quad (1)$$

It is known, in the context of *information geometry*¹, that a “reasonable” statistical manifold S possesses a uniquely defined *dualistic structure*² $(h_F, \nabla^{(e)}, \nabla^{(m)})$; the metric h_F is called the *Fisher metric*, $\nabla^{(e)}$ is the *exponential connection* and $\nabla^{(m)}$ is the *mixture connection*. These geometrical objects encode many important statistical properties of the statistical manifold S . For example, they can be used to give lower bounds in estimation problems (e.g. Cramér-Rao inequality, see [AN00, MR93]).

The Fisher metric and exponential connection are defined as follows. For a chart $\xi = (\xi_1, \dots, \xi_n)$ of S , and denoting Γ_{ij}^k the Christoffel symbols of $\nabla^{(e)}$ in this chart, we have :

¹Information geometry is a branch of statistics characterized by its use of differential geometric techniques, see [AN00, MR93].

²A dualistic structure on a manifold M is a triple (g, ∇, ∇^*) , where g is a Riemannian metric and where ∇, ∇^* are affine connections which are dual to each other in the sense that $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X^* Z)$ for all vector fields X, Y, Z on M . The connection ∇^* is called the *dual connection* of ∇ (and vice versa).

- $(h_F)_\xi(\partial_i, \partial_j) := E_{p_\xi}(\partial_i \ln(p_\xi) \cdot \partial_j \ln(p_\xi))$,
- $\Gamma_{ij}^k(\xi) := E_{p_\xi} \left[\left(\partial_i \partial_j \ln(p_\xi) \cdot \partial_k \ln(p_\xi) \right) \partial_k \ln(p_\xi) \right]$,

where E_{p_ξ} denotes the mean, or expectation, with respect to the probability $p_\xi dx$ (here p_ξ denotes the unique probability density function determined by ξ), and where ∂_i is a shorthand for $\partial/\partial \xi_i$. The connection $\nabla^{(m)}$ is obtained via the duality between $\nabla^{(m)}$ and $\nabla^{(e)}$.

It has recently been stressed in [Mol12, Mol] that dualistic structures on statistical manifolds play a central role in the mathematical foundations of *finite dimensional*³ quantum mechanics, in which *Dombrowski's construction* is particularly important. Recall that given a metric g and a connection ∇ on a manifold M (∇ needs not be the Levi-Civita connection), Dombrowski's construction yields an almost Hermitian structure $(g^{TM}, J^{TM}, \omega^{TM})$ on TM , the latter structure being Kähler if and only if ∇ and ∇^* are both flat (see [Dom62, Mol]). For example, if $\Omega := \{x_1, \dots, x_n\}$ is a finite set and if \mathcal{P}_n^\times is the statistical manifold of nowhere vanishing probabilities $p : \Omega \rightarrow \mathbb{R}$, $p > 0$, $\sum_{k=1}^n p(x_k) = 1$, then $\nabla^{(e)}$ and $\nabla^{(m)}$ are flat and the Kähler structure associated to $(h_F, \nabla^{(e)})$ via Dombrowski's construction on $T\mathcal{P}_n^\times$ is locally isomorphic to the complex projective space $\mathbb{P}(\mathbb{C}^n)$ of complex lines in \mathbb{C}^n [Mol12].

This example, although mathematically simple, is physically fundamental. For, as it is known, a finite dimensional quantum system can be entirely described by the Kähler structure of the complex projective space $\mathbb{P}(\mathbb{C}^n)$ associated to the Hilbert space \mathbb{C}^n of quantum states⁴; this is the so-called *geometrical formulation* of quantum mechanics (see for example [AS99]). Hence, by realizing an open dense subset of $\mathbb{P}(\mathbb{C}^n)$ as a “Kählerification” of \mathcal{P}_n^\times via Dombrowski's construction (see [Mol] for a precise statement), we directly connect information geometry to quantum mechanics.

Based on these observations, and together with other mathematical results, we were led in [Mol] to conjecture that the quantum formalism, at least in finite dimension, has a purely information-theoretical origin in which the Fisher metric and the exponential connection, together with Dombrowski's construction, are crucial.

The purpose of the present paper is to investigate the properties of a particular *infinite dimensional* quantum system in the light of the results obtained in [Mol12, Mol]. Our quantum system is a non-relativistic quantum particle, mathematically represented by a wave function $\psi : M \rightarrow \mathbb{C}$, living on a compact and connected Riemannian manifold (M, g) , and whose dynamics is governed by the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2} \Delta \psi + V\psi, \quad (2)$$

where \hbar is Planck constant, Δ is the Laplacian operator and where $V : M \rightarrow \mathbb{R}$ is a given potential.

To this system, we attach, as a statistical model, the space \mathcal{D} of smooth density probability functions on M :

$$\mathcal{D} := \left\{ \rho \in C^\infty(M, \mathbb{R}) \mid \rho > 0, \int_M \rho d\text{vol}_g = 1 \right\}, \quad (3)$$

where $d\text{vol}_g$ denotes the Riemannian volume form associated to g (M is assumed oriented). We regard the space \mathcal{D} as an infinite dimensional analogue of \mathcal{P}_n^\times .

³By finite dimensional, we are referring to quantum systems whose associated Hilbert spaces \mathcal{H} are finite dimensional, like for the spin of a particle.

⁴This is also true for infinite dimensional quantum systems.

In this paper, we show the following: first, that it is possible to rewrite the Schrödinger equation (2) into a genuine system of Lagrangian equations on $T\mathcal{D}$ for an appropriate Lagrangian $\mathcal{L} : T\mathcal{D} \rightarrow \mathbb{R}$. Second, that this Lagrangian system can be reformulated in a symplectic way on $T\mathcal{D}$ using geometric mechanical methods. Finally, that the corresponding symplectic form $\Omega_{\mathcal{L}}$ on $T\mathcal{D}$ is nothing but the fundamental form of the almost Hermitian structure associated, via Dombrowski's construction, to a natural metric $g^{\mathcal{D}}$ and a connection $\nabla^{\mathcal{D}}$ living on \mathcal{D} .

The couple $(g^{\mathcal{D}}, \nabla^{\mathcal{D}})$ on \mathcal{D} is thus –and this is the main observation of this paper– an infinite dimensional analogue of $(h_F, \nabla^{(e)})$ on \mathcal{P}_n^{\times} which encodes the dynamics of the quantum particle, exactly as in the finite dimensional case (see [Mol]).

Additionally, we observe that the almost complex structure of $T\mathcal{D}$ is not integrable and that, contrary to $\nabla^{(e)}$, the connection $\nabla^{\mathcal{D}}$ on \mathcal{D} has a non-trivial torsion (this proves in particular that $\nabla^{\mathcal{D}}$ is *not* the Levi-Civita connection associated to $g^{\mathcal{D}}$).

This paper is organized as follows. In §2 we describe the geometry of \mathcal{D} and its tangent bundle; that will allow us, in §3 and §4, to recast the Schrödinger equation directly on $T\mathcal{D}$, in a Lagrangian form (§3) and in a Hamiltonian form (§4). Finally, in §5 we observe that the symplectic form $\Omega_{\mathcal{L}}$ on $T\mathcal{D}$ describing the dynamics of the quantum particle is nothing but the fundamental form of the almost Hermitian structure associated to $(g^{\mathcal{D}}, \nabla^{\mathcal{D}})$ on \mathcal{D} . The paper ends with §6 where we discuss a possible definition for the wave function associated to a moving probability. An example is considered.

Some of our results are expressed in the category of *tame Fréchet manifolds* introduced by Hamilton in [Ham82]. The relevant definitions are recalled in §1.

1 Hamilton's category of tame Fréchet manifolds

In this section, we review very briefly the category of tame Fréchet manifolds introduced by Hamilton in [Ham82].

Definition 1.1. 1. A *graded Fréchet space* $(F, \{\|\cdot\|_n\}_{n \in \mathbb{N}})$, is a Fréchet space F whose topology is defined by a collection of seminorms $\{\|\cdot\|_n\}_{n \in \mathbb{N}}$ which are increasing in strength:

$$\|x\|_0 \leq \|x\|_1 \leq \|x\|_2 \leq \dots \quad (4)$$

for all $x \in F$.

2. A linear map $L : F \rightarrow G$ between two graded Fréchet spaces F and G is *tame* (of degree r and base b) if for all $n \geq b$, there exists a constant $C_n > 0$ such that for all $x \in F$,

$$\|L(x)\|_n \leq C_n \|x\|_{n+r}. \quad (5)$$

3. If $(B, \|\cdot\|_B)$ is a Banach space, then $\Sigma(B)$ denotes the graded Fréchet space of all sequences $\{x_k\}_{k \in \mathbb{N}}$ of B such that for all $n \geq 0$,

$$\|\{x_k\}_{k \in \mathbb{N}}\|_n := \sum_{k=0}^{\infty} e^{nk} \|x_k\|_B < \infty. \quad (6)$$

4. A graded Fréchet space F is *tame* if there exist a Banach space B and two tame linear maps $i : F \rightarrow \Sigma(B)$ and $p : \Sigma(B) \rightarrow F$ such that $p \circ i$ is the identity on F .

5. Let F, G be two tame Fréchet spaces, U an open subset of F and $f : U \rightarrow G$ a map. We say that f is a smooth tame map if f is smooth⁵ and if for every $k \in \mathbb{N}$ and for every $(x, u_1, \dots, u_k) \in U \times F \times \dots \times F$, there exist a neighborhood V of (x, u_1, \dots, u_k) in $U \times F \times \dots \times F$ and $b_k, r_0, \dots, r_k \in \mathbb{N}$ such that for every $n \geq b_k$, there exists $C_{k,n}^V > 0$ such that

$$\|d^k f(y)\{v_1, \dots, v_k\}\|_n \leq C_{k,n}^V (1 + \|y\|_{n+r_0} + \|v_1\|_{n+r_1} + \dots + \|v_k\|_{n+r_k}), \quad (7)$$

for every $(y, v_1, \dots, v_k) \in V$, where $d^k f : U \times F \times \dots \times F \rightarrow G$ denotes the k th derivative of f .

Remark 1.2. In this paper, we use interchangeably the notation $(df)(x)\{v\}$ or $f_{*x}v$ for the first derivative of f at a point x in direction v .

As one may notice, tame Fréchet spaces and smooth tame maps form a category, and it is thus natural to define a tame Fréchet manifold as a Hausdorff topological space with an atlas of coordinates charts taking their value in tame Fréchet spaces, such that the coordinate transition functions are all smooth tame maps (see [Ham82]). The definition of a tame smooth map between tame Fréchet manifolds is then straightforward, and we thus obtain a subcategory of the category of Fréchet manifolds.

In order to avoid confusion, let us also make precise our notion of submanifold. We will say that a subset \mathcal{M} of a tame Fréchet manifold \mathcal{M} , endowed with the trace topology, is a submanifold, if for every point $x \in \mathcal{M}$, there exists a chart (\mathcal{U}, φ) of \mathcal{M} such that $x \in \mathcal{U}$ and such that $\varphi(\mathcal{U} \cap \mathcal{M}) = U \times \{0\}$, where $\varphi(\mathcal{U}) = U \times V$ is a product of two open subsets of tame Fréchet spaces. Note that a submanifold of a tame Fréchet manifold is also a tame Fréchet manifold.

For the sake of completeness, let us state here the *raison d'être* of tame Fréchet spaces and tame Fréchet manifolds (see [Ham82]) :

Theorem 1.3 (Nash-Moser inverse function Theorem). *Let F, G be two tame Fréchet spaces, U an open subset of F and $f : U \rightarrow G$ a smooth tame map. If there exists an open subset $V \subseteq U$ such that*

1. $df(x) : F \rightarrow G$ is an linear isomorphism for all $x \in V$,
2. the map $V \times G \rightarrow F$, $(x, v) \mapsto (df(x))^{-1}\{v\}$ is a smooth tame map,

then f is locally invertible on V and each local inverse is a smooth tame map.

Remark 1.4. The Nash-Moser inverse function Theorem is important in geometric hydrodynamics, since one of its most important geometric objects, namely the group of all smooth volume preserving diffeomorphisms $\text{SDiff}_\mu(M) := \{\varphi \in \text{Diff}(M) \mid \varphi^* \mu = \mu\}$ of an oriented manifold (M, μ) , can only be given a rigorous Fréchet Lie group structure by using an inverse function theorem (at least up to now). To our knowledge, only two authors succeeded in doing this. The first was Omori who showed and used an inverse function theorem in terms of ILB-spaces (“inverse limit of Banach spaces”, see [Omo97]), and later on, Hamilton with his category of tame Fréchet spaces together with the Nash-Moser inverse function Theorem (see [Ham82]). Nowadays, it is nevertheless not uncommon to find mistakes or big gaps in the literature when it comes to the differentiable structure of $\text{SDiff}_\mu(M)$, even in some specialized textbooks in infinite dimensional geometry. The case of M being non-compact is even worse, and of course, no proof that $\text{SDiff}_\mu(M)$ is a “Lie group” is available in this case.

⁵By smooth we mean that $f : U \subseteq F \rightarrow G$ is continuous and that for all $k \in \mathbb{N}$, the k th derivative $d^k f : U \times F \times \dots \times F \rightarrow G$ exists and is jointly continuous on the product space, such as described in [Ham82].

Finally, and quite apart from the category of Hamilton, let us remind one of the most useful result of the convenient calculus (see [KM97]) :

Lemma 1.5. *Let F, G be two Fréchet spaces, U an open subset of F and $f : U \rightarrow G$ a map. Then f is smooth in the sense of Hamilton (see footnote 5), if and only if $f \circ c : I \rightarrow G$ is a smooth curve in G whenever $c : I \rightarrow U$ is a smooth curve in U .*

As one may show, if M, N are manifolds, M being compact, then a smooth curve in the Fréchet manifold $C^\infty(M, N)$ may be identified with a smooth map $f : I \times M \rightarrow N$, its time derivative being identified with the partial derivative of f with respect to t . From this together with Lemma 1.5, it is usually easy to show that a map defined between submanifolds of spaces of maps is smooth : it suffices to compose this map with a smooth curve, and then to check that the result is smooth in the “finite dimensional sense” with respect to all the “finite dimensional” variables (see [KM97]).

2 The manifold structure of \mathcal{D} and its tangent bundle

Let (M, g) be a compact, connected and oriented Riemannian manifold with Riemannian volume form $d\text{vol}_g$, and let \mathcal{D} be the space of smooth density probability functions on M :

$$\mathcal{D} := \{ \rho \in C^\infty(M, \mathbb{R}) \mid \rho > 0, \int_M \rho d\text{vol}_g = 1 \}. \quad (8)$$

Throughout this section, we shall write $C^\infty(M)$ instead of $C^\infty(M, \mathbb{R})$ (and similar for subspaces of $C^\infty(M, \mathbb{R})$) if there is no danger of confusion. We shall also use the notation $\mathbb{R}_+^* := \{ r \in \mathbb{R} \mid r > 0 \}$.

Let us start with the differentiable structure of \mathcal{D} .

Proposition 2.1. *The space \mathcal{D} is a tame Fréchet submanifold of the tame Fréchet space $C^\infty(M)$, and for $\rho \in \mathcal{D}$,*

$$T_\rho \mathcal{D} \cong C_0^\infty(M), \quad (9)$$

where

$$C_0^\infty(M) := \{ f \in C^\infty(M) \mid \int_M f d\text{vol}_g = 0 \}. \quad (10)$$

Observe that we have the following L^2 -orthogonal decomposition,

$$C^\infty(M) = C_0^\infty(M) \oplus \mathbb{R}, \quad (11)$$

the decomposition being given, for $f \in C^\infty(M)$, by

$$f = f - \frac{1}{\text{Vol}(M)} \int_M f \cdot d\text{vol}_g + \frac{1}{\text{Vol}(M)} \int_M f \cdot d\text{vol}_g, \quad (12)$$

where $\text{Vol}(M) := \int_M d\text{vol}_g$ denotes the Riemannian volume of M . In particular, the space $C_0^\infty(M)$ is a tame Fréchet space (it is a Fréchet space because $C_0^\infty(M)$ is closed in $C^\infty(M)$ and it is also a tame space because $C^\infty(M)$ is tame, see [Ham82], Definition 1.3.1 and Corollary 1.3.9).

Proof of Proposition 2.1. The proof relies on the following tame diffeomorphism of tame Fréchet manifolds:

$$\Phi : \begin{cases} C^\infty(M) \rightarrow C_0^\infty(M) \times \mathbb{R}, \\ f \mapsto \left(f - \text{Vol}(M)^{-1} \int_M f \, d\text{vol}_g, \text{Vol}(M)^{-1} \int_M f \, d\text{vol}_g - \text{Vol}(M)^{-1} \right). \end{cases} \quad (13)$$

Using Φ , it is possible to define splitting charts for $C^\infty(M)$; indeed, the space $C^\infty(M, \mathbb{R}_+^*)$ being clearly an open subset of $C^\infty(M)$ for its natural Fréchet space topology, every $\rho \in \mathcal{D}$ possesses an open neighborhood in $C^\infty(M)$, say U_ρ , such that $\rho \in U_\rho \subseteq C^\infty(M, \mathbb{R}_+^*)$, and, restricting U_ρ if necessary, we may assume that $\Phi(U_\rho) = V_\rho \times W_\rho$ where V_ρ and W_ρ are open subsets of $C_0^\infty(M)$ and \mathbb{R} respectively. But now, $(U_\rho, \Phi|_{U_\rho})$ is a chart of $C^\infty(M)$ and it is easy to see that

$$(\Phi|_{U_\rho})(U_\rho \cap \mathcal{D}) = V_\rho \times \{0\}. \quad (14)$$

The proposition follows. \square

We now want to give a geometrical description of the tangent space of \mathcal{D} . Recall that if $X \in \mathfrak{X}(M)$ is a vector field on M , then its divergence with respect to the volume form $d\text{vol}_g$ is the unique function $\text{div}(X) : M \rightarrow \mathbb{R}$ satisfying $\mathcal{L}_X(d\text{vol}_g) = \text{div}(X) \cdot d\text{vol}_g$, \mathcal{L}_X being the Lie derivative in direction X . Using the divergence operator, we define, for $f : M \rightarrow \mathbb{R}_+^*$, an elliptic differential operator $P_f : C^\infty(M) \rightarrow C^\infty(M)$ via the formula

$$P_f(u) := \text{div}(f \cdot \nabla u), \quad (15)$$

where $u : M \rightarrow \mathbb{R}$ is a smooth function. Observe that

- $P_1 = \Delta$ is the Laplacian operator,
- P_f takes values in $C_0^\infty(M)$ since the integral with respect to the Riemannian volume form of a divergence is always zero by application of Stokes' Theorem.
- The kernel of P_f reduces to the constant functions. This is due to the fact that P_f is a second order elliptic differential operator whose constant term $P_f(1)$ is zero, and it is well known that for such differential operators on compact manifolds, the kernel reduces to the constant functions (see [Jos02]).

Lemma 2.2. *For $f \in C^\infty(M)$, $f > 0$, the restriction \bar{P}_f of the operator P_f to $C_0^\infty(M)$,*

$$\bar{P}_f : C_0^\infty(M) \rightarrow C_0^\infty(M), \quad (16)$$

is an isomorphism of Fréchet spaces. Moreover, its family of inverses

$$C^\infty(M, \mathbb{R}_+^*) \times C_0^\infty(M) \rightarrow C_0^\infty(M), \quad (f, h) \mapsto (\bar{P}_f)^{-1}(h) \quad (17)$$

forms a smooth tame map.

Proof. The operator \bar{P}_f is injective since its kernel is the intersection of the kernel of P_f with the space $C_0^\infty(M)$, which is zero.

For the surjectivity, take $\tilde{f} : [0, 1] \rightarrow C^\infty(M, \mathbb{R}_+^*)$ a continuous path such that $\tilde{f}_0 \equiv 1$ and $\tilde{f}_1 = f$. As one may see, $P_{\tilde{f}_t}$ defines a continuous path of elliptic operators (acting on a suitable Sobolev space), and

by the topological invariance of the analytic index Ind of an elliptic operator together with the fact that the analytic index of $\Delta : C^\infty(M) \rightarrow C^\infty(M)$ is zero, we have :

$$\text{Ind}(P_f) = \text{Ind}(P_{\tilde{f}_1}) = \text{Ind}(P_{\tilde{f}_0}) = \text{Ind}(\Delta) = 0. \quad (18)$$

Hence, the codimension of the image $\text{Im}(P_f)$ of P_f is 1, and since $\text{Im}(P_f) \subseteq C_0^\infty(M)$, this later space being of codimension 1, $\text{Im}(P_f) = C_0^\infty(M)$. It follows that $\overline{P}_f : C_0^\infty(M) \rightarrow C_0^\infty(M)$ is a bijection.

Finally, \overline{P}_f is continuous since it is a differential operator, and its inverse is also continuous by application of the open mapping Theorem.

The fact that the family of inverses defined in (17) forms a smooth tame map is a consequence of a result due to Hamilton (see [Ham82], Theorem 3.3.3) about the family of inverses of a family of invertible (up to something of finite dimension) elliptic differential operators, applied to the following map :

$$\begin{cases} C^\infty(M, \mathbb{R}_+^*) \times C^\infty(M) \times \mathbb{R} \rightarrow C^\infty(M) \times \mathbb{R}, \\ (f, h, x) \mapsto (P_f(h) + x, \int_M h \, \text{dvol}_g). \end{cases} \quad (19)$$

The result of Hamilton implies the existence of a smooth Green operator $G : C^\infty(M, \mathbb{R}_+^*) \times C^\infty(M) \rightarrow C_0^\infty(M)$ whose restriction to $C^\infty(M, \mathbb{R}_+^*) \times C_0^\infty(M)$ coincides with the family considered in (17). The lemma follows. \square

Proposition 2.3. *Let $X \in \mathfrak{X}(M)$ be a vector field and let $\rho \in \mathcal{D}$ be a smooth density. For $h \in T_\rho \mathcal{D} \cong C_0^\infty(M)$, there exists a unique function $\phi : M \rightarrow \mathbb{R}$ (defined up to an additive constant), such that*

$$h = \text{div}(\rho(\nabla\phi + X)). \quad (20)$$

Moreover, the map

$$T\mathcal{D} \rightarrow \mathcal{D} \times \nabla C^\infty(M), \quad h = \text{div}(\rho(\nabla\phi + X)) \mapsto (\rho, \nabla\phi), \quad (21)$$

is a non-linear tame isomorphism of tame Fréchet vector bundles, $\mathcal{D} \times \nabla C^\infty(M)$ being the trivial vector bundle over \mathcal{D} .

Proof. For $\rho \in \mathcal{D}$ and $h \in T_\rho \mathcal{D} \cong C_0^\infty(M)$, define $\phi \in C_0^\infty(M)$ by letting

$$\phi := (\overline{P}_\rho)^{-1}[h - \text{div}(\rho X)] \quad (22)$$

(note that $\text{div}(\rho X) \in C_0^\infty(M)$, and thus $h - \text{div}(\rho X) \in C_0^\infty(M)$).

By applying the operator \overline{P}_ρ to (22), we see that

$$\begin{aligned} \overline{P}_\rho(\phi) = h - \text{div}(\rho X) &\Rightarrow \text{div}(\rho \nabla\phi) = h - \text{div}(\rho X) \\ &\Rightarrow h = \text{div}(\rho(\nabla\phi + X)). \end{aligned} \quad (23)$$

Moreover, if $\phi' : M \rightarrow \mathbb{R}$ satisfies $h = \text{div}(\rho(\nabla\phi' + X))$, then $P_\rho(\phi - \phi') = 0$, and thus, $\phi - \phi'$ is a constant function. The first assertion of the proposition follows.

For the second assertion, it is clear that the map defined in (21) is a fiber preserving bijection; its smoothness is a consequence of the smoothness of the family of inverses (17) (that one may apply in charts such as defined in the proof of Proposition 2.1, or directly using the convenient calculus developed in [KM97]); this map is also tame for the same reason and its inverse is clearly a smooth tame map. The proposition follows. \square

Remark 2.4. The space of all gradients $\nabla C^\infty(M)$ is a tame Fréchet space. This comes from the fact that the Helmholtz-Hodge decomposition

$$\mathfrak{X}(M) = \mathfrak{X}_{\text{dvol}_g}(M) \oplus \nabla C^\infty(M), \quad (24)$$

where $\mathfrak{X}_{\text{dvol}_g}(M) := \{X \in \mathfrak{X}(M) \mid \text{div}(X) = 0\}$, is a topological direct sum (see [Ham82]). As a consequence, the space $\mathcal{D} \times \nabla C^\infty(M)$ is a tame Fréchet space, and in particular, it is a trivial tame Fréchet vector bundle over \mathcal{D} .

Remark 2.5. In connection with electromagnetism, if we allow the vector field $X \in \mathfrak{X}(M)$ of Proposition 2.3 to be time-dependent, then an obvious modification of the proof of Proposition 2.3 shows that the map

$$\begin{cases} T\mathcal{D} \times \mathbb{R} \rightarrow \mathcal{D} \times \nabla C^\infty(M) \times \mathbb{R}, \\ (\rho, h = \text{div}(\rho(\nabla\phi_t + X_t)), t) \mapsto (\rho, \nabla\phi_t, t), \end{cases}$$

is a smooth tame diffeomorphism.

Remark 2.6. In §1 and §2 we were working in the category of tame Fréchet spaces, but in the sequel we will relax this hypothesis and simply work with the usual Fréchet category.

3 Euler-Lagrange equations on \mathcal{D} and the Schrödinger equation

Having a precise and geometric description of the tangent bundle of \mathcal{D} , it is easy to write interesting Lagrangians on \mathcal{D} . Indeed, for a time-dependent vector field $X_t \in \mathfrak{X}(M)$, a time-dependent potential $V_t : M \rightarrow \mathbb{R}$, and using the diffeomorphism $T\mathcal{D} \times \mathbb{R} \rightarrow \mathcal{D} \times \nabla C^\infty(M) \times \mathbb{R}$ of Remark 2.5, we can consider, with an abuse of notation, the following time-dependent Lagrangian :

$$\begin{aligned} \mathcal{L}(\rho, h = \text{div}(\rho(\nabla\phi_t + X_t)), t) &= \mathcal{L}(\rho, \nabla\phi_t, t) := \\ &\int_M \left(\frac{1}{2} \|\nabla\phi_t\|^2 - \|X_t\|^2 - V_t \right) \rho \cdot \text{dvol}_g - \frac{\hbar^2}{2} \int_M \|\nabla(\sqrt{\rho})\|^2 \cdot \text{dvol}_g. \end{aligned} \quad (25)$$

Note that \mathcal{L} is smooth by application of the convenient calculus together with Remark 2.5.

By using the formula

$$\frac{1}{4} \frac{\|\nabla u\|^2}{u^2} - \frac{1}{2} \frac{\Delta u}{u} = -\frac{\Delta(\sqrt{u})}{\sqrt{u}}, \quad (26)$$

which is valid for every smooth function $u : M \rightarrow \mathbb{R}$, and by doing an usual fixed end-point variation of the Lagrangian \mathcal{L} , one easily finds the following Euler-Lagrange equations :

Proposition 3.1. The Euler-Lagrange equations associated to the Lagrangian \mathcal{L} defined in (25), are given by

$$\begin{cases} \frac{\partial \phi}{\partial t} &= \frac{1}{2} \|\nabla\phi + X\|^2 + V - \frac{\hbar^2}{2} \frac{\Delta(\sqrt{\rho})}{\sqrt{\rho}} + c_t, \\ \frac{\partial \rho}{\partial t} &= \text{div}(\rho(\nabla\phi + X)), \end{cases} \quad (27)$$

where $\rho : I \subseteq \mathbb{R} \rightarrow \mathcal{D}$ is a smooth curve in \mathcal{D} and where c_t is a time-dependent constant.

Remark 3.2. The second equation in (27) has actually nothing to do with variational principles; its is only the geometric way to express tangent vectors in \mathcal{D} , such as described in Proposition 2.3.

Remark 3.3. Due to similarities with equations of hydrodynamical type, the system of equations (27) is sometimes referred to as the hydrodynamical formulation of quantum mechanics.

Note that the appearance of the time-dependant constant c_t in (27) is due to the L^2 -orthogonal decomposition (11).

Remark 3.4. By doing the change of variable $\phi' := \phi - \int c_t dt$ if necessary, one may assume that the time-dependant constant c_t of Proposition 3.1 is zero.

As it is well known, if $c_t \equiv 0$, then the system (27) is equivalent to the Schrödinger equation for a quantum charged particle in an electromagnetic field:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2} \Delta \psi - \frac{\hbar}{i} g(X, \nabla \phi) + \frac{1}{2} \left(-\frac{\hbar}{i} \operatorname{div}(X) + \|X\|^2 \right) \psi + V\psi, \quad (28)$$

where

$$\psi := \sqrt{\rho} e^{-\frac{i}{\hbar} \phi}. \quad (29)$$

Using Remark 3.4, we can thus state the following corollary.

Corollary 3.5. Let ρ be a solution in \mathcal{D} of the Euler-Lagrange equations associated to the Lagrangian $\mathcal{L} : T\mathcal{D} \rightarrow \mathbb{R}$ (see (25)), with $\partial \rho / \partial t = \operatorname{div}(\rho(\nabla \phi + X))$. Then the wave function associated to ρ ,

$$\psi := \sqrt{\rho} e^{-\frac{i}{\hbar} \left(\phi - \int c_t dt \right)}, \quad (30)$$

(see (27) for the definition of c_t), satisfies the Schrödinger equation (28).

Remark 3.6. For a smooth function $\psi : M \rightarrow \mathbb{C}$, let us denote by $[\psi]$ the complex line generated by ψ in the complex Hilbert space $\mathcal{H} := L^2(M, \mathbb{C})$ (the latter being endowed with its natural L^2 -scalar product). Let us also consider the following map

$$T : T\mathcal{D} \rightarrow \mathbb{P}(\mathcal{H}), \quad (\rho, \nabla \phi) \mapsto [\sqrt{\rho} e^{-\frac{i}{\hbar} \phi}], \quad (31)$$

where $\mathbb{P}(\mathcal{H})$ denotes the complex projective space of complex lines in \mathcal{H} . As one may easily see, this map is well defined, and since

$$[\sqrt{\rho} e^{-\frac{i}{\hbar} \left(\phi - \int c_t dt \right)}] = [\sqrt{\rho} e^{-\frac{i}{\hbar} \phi}], \quad (32)$$

Corollary 3.5 implies that T maps solutions of the Euler-Lagrange equations (27) to solutions of the Schrödinger equation (28), projected on $\mathbb{P}(\mathcal{H})$.

4 Hamiltonian formulation

In this section, we continue our study of the dynamics of a quantum particle initiated in §3, but we will now focus on the Hamiltonian formulation.

We will still assume that (M, g) is a compact, connected and oriented Riemannian manifold, but for simplicity, we will assume that the particle is only under the influence of a time-independent potential $V : M \rightarrow \mathbb{R}$.

Usually, the Hamiltonian formulation of a Lagrangian system is obtained by pulling back the canonical symplectic form of the cotangent bundle of the configuration manifold via the Legendre transform (see [AM78]). In our case, the configuration manifold \mathcal{D} being infinite dimensional, its cotangent bundle is no more a Fréchet manifold, rather a manifold modelled on more general locally convex topological spaces, and we thus want to avoid it.

To this end, we observe (see Corollary 3.5) that the Lagrangian which describes a quantum particle under the influence of a potential is given by

$$\mathcal{L}(\rho, \nabla\phi) = \frac{1}{2} \int_M \|\nabla\phi\|^2 \rho \cdot d\text{vol}_g - \int_M V \rho \cdot d\text{vol}_g - \frac{\hbar^2}{2} \int_M \|\nabla(\sqrt{\rho})\|^2 \cdot d\text{vol}_g, \quad (33)$$

where $(\rho, \nabla\phi) \in \mathcal{D} \times \nabla C^\infty(M)$.

This Lagrangian is of the form kinetic energy minus two potential terms, and, heuristically at least, this implies that the associated Lagrangian symplectic form⁶ on $T\mathcal{D}$ is uniquely determined by the metric

$$(g^{\mathcal{D}})_\rho((\rho, \nabla\phi), (\rho, \nabla\phi')) := \int_M g(\nabla\phi, \nabla\phi') \rho \cdot d\text{vol}_g, \quad (34)$$

since potentials vanish under the Legendre transform. This motivates us, by mimicking the finite dimensional construction, to define the canonical 1-form $\Theta_{\mathcal{L}}$ on $T\mathcal{D} \cong \mathcal{D} \times \nabla C^\infty(M)$ via the formula :

$$(\Theta_{\mathcal{L}})_{(\rho, \nabla\phi)}(A_{(\rho, \nabla\phi)}) := (g^{\mathcal{D}})_\rho\left(\nabla\phi, (\pi^{T\mathcal{D}})_{*(\rho, \nabla\phi)} A_{(\rho, \nabla\phi)}\right), \quad (35)$$

where $A_{(\rho, \nabla\phi)} \in T_{(\rho, \nabla\phi)}T\mathcal{D}$ and where $\pi^{T\mathcal{D}} : T\mathcal{D} \rightarrow \mathcal{D}$ denotes the canonical projection.

The right hand side of (35) is formally the pull back of the canonical 1-form of the full cotangent bundle $T^*\mathcal{D}$ via the Legendre transform associated to \mathcal{L} . We will not explain this point any further, but we will consider (35) as the starting point of our study of the Hamiltonian description of a quantum particle.

Our aim is now to compute explicitly the differential of $\Theta_{\mathcal{L}}$, and to show that $\Omega_{\mathcal{L}} := -d\Theta_{\mathcal{L}}$ is a symplectic form on $T\mathcal{D}$. To this end, we will use the following identification

$$T(T\mathcal{D}) \cong \mathcal{D} \times \nabla C^\infty(M) \times \nabla C^\infty(M) \times \nabla C^\infty(M), \quad (36)$$

the diffeomorphism being given by

$$\left. \frac{d}{dt} \right|_0 (\rho_t, \nabla\phi + t\nabla\psi_2) \mapsto (\rho_0, \nabla\phi, \nabla\psi_1, \nabla\psi_2), \quad (37)$$

where ρ_t is a smooth curve in \mathcal{D} satisfying

$$\left. \frac{d}{dt} \right|_0 \rho_t = \text{div}(\rho_0 \cdot \nabla\psi_1). \quad (38)$$

⁶Recall that the Lagrangian symplectic form $\omega_L \in \Omega^2(TM)$ associated to a Lagrangian $L : TM \rightarrow \mathbb{R}$, is the pull back via the Legendre transform $\mathbb{F}L : TM \rightarrow T^*M$, $\mathbb{F}L(u_x)(v_x) := \left. \frac{d}{dt} \right|_0 L(u_x + tv_x)$, of the canonical symplectic form $\omega = -d\theta$ of the cotangent bundle, where the canonical 1-form $\theta \in \Omega^1(T^*M)$ is defined, for $A_{\alpha_x} \in T_{\alpha_x}T^*M$, by $\theta_{\alpha_x}(A_{\alpha_x}) := \alpha_x((\pi^{T^*M})_{*\alpha_x} A_{\alpha_x})$, $\pi^{T^*M} : T^*M \rightarrow M$ being the canonical projection, see [AM78].

Using (34) and (36), it is clear that (35) may be rewritten

$$(\Theta_{\mathcal{L}})_{(\rho, \nabla \phi)}(\rho, \nabla \phi, \nabla \psi_1, \nabla \psi_2) = \int_M g(\nabla \phi, \nabla \psi_1) \rho \cdot d\text{vol}_g. \quad (39)$$

Our strategy to compute the differential of $\Theta_{\mathcal{L}}$ at a point $(\rho, \nabla \phi)$, will be to use the formula

$$(d\Theta_{\mathcal{L}})_{(\rho, \nabla \phi)}(X, Y) = X_{(\rho, \nabla \phi)}(\Theta_{\mathcal{L}}(Y)) - Y_{(\rho, \nabla \phi)}(\Theta_{\mathcal{L}}(X)) - (\Theta_{\mathcal{L}})_{(\rho, \nabla \phi)}([X, Y]), \quad (40)$$

where X, Y are vector fields on $T\mathcal{D}$.

As the above formula is tensorial in X and Y , we are free to choose X and Y arbitrary at a given point $(\rho, \nabla \phi)$, and to extend these vector fields as simply as possible elsewhere. A natural choice is to set, for any $(\rho, \nabla \phi) \in T\mathcal{D}$,

$$X_{(\rho, \nabla \phi)} := (\rho, \nabla \phi, \nabla \psi_1, \nabla \psi_2) \quad \text{and} \quad Y_{(\rho, \nabla \phi)} := (\rho, \nabla \phi, \nabla \alpha_1, \nabla \alpha_2), \quad (41)$$

where $\nabla \psi_1, \nabla \psi_2, \nabla \alpha_1, \nabla \alpha_2$ are held fixed.

In view of (40), we now have to compute $X_{(\rho, \nabla \phi)}(\Theta_{\mathcal{L}}(Y))$ and $(\Theta_{\mathcal{L}})_{(\rho, \nabla \phi)}([X, Y])$, with X and Y as defined in (41).

Lemma 4.1. *We have :*

$$X_{(\rho, \nabla \phi)}(\Theta_{\mathcal{L}}(Y)) = \int_M g(\nabla \phi, \nabla \alpha_1) \text{div}(\rho \cdot \nabla \psi_1) \cdot d\text{vol}_g + \int_M g(\nabla \psi_2, \nabla \alpha_1) \rho \cdot d\text{vol}_g. \quad (42)$$

Proof. Let ρ_t be a curve in \mathcal{D} satisfying

$$\rho_0 = \rho \quad \text{and} \quad \frac{\partial \rho_t}{\partial t} = \text{div}(\rho_t \cdot \nabla \psi_1). \quad (43)$$

If $c(t) := (\rho_t, \nabla \phi + t \nabla \psi_2)$, then

$$\dot{c}(0) = (\rho, \nabla \phi, \nabla \psi_1, \nabla \psi_2) = X_{(\rho, \nabla \phi)}, \quad (44)$$

and thus,

$$\begin{aligned} & X_{(\rho, \nabla \phi)}(\Theta_{\mathcal{L}}(Y)) \\ &= \left. \frac{d}{dt} \right|_0 (\Theta_{\mathcal{L}})_{c(t)}(Y_{c(t)}) = \left. \frac{d}{dt} \right|_0 \Theta_{\mathcal{L}}(\rho_t, \nabla \phi + t \nabla \psi_2, \nabla \alpha_1, \nabla \alpha_2) \\ &= \left. \frac{d}{dt} \right|_0 \left[\int_M g(\nabla \phi, \nabla \alpha_1) \rho_t \cdot d\text{vol}_g + t \int_M g(\nabla \psi_2, \nabla \alpha_1) \rho_t \cdot d\text{vol}_g \right] \\ &= \int_M g(\nabla \phi, \nabla \alpha_1) \text{div}(\rho \cdot \nabla \psi_1) \cdot d\text{vol}_g + \int_M g(\nabla \psi_2, \nabla \alpha_1) \rho \cdot d\text{vol}_g. \end{aligned} \quad (45)$$

The lemma follows. \square

For the term $(\Theta_{\mathcal{L}})_{(\rho, \nabla \phi)}([X, Y])$, we need to compute the Lie bracket $[X, Y]$ of X and Y , and this can be done with a good description of the flow φ_t^X of X .

This description may be obtained with the following map

$$D : \text{Diff}(M) \rightarrow C^\infty(M), \quad (46)$$

which is defined, for a φ belonging to the group of all diffeomorphisms $\text{Diff}(M)$, via the formula

$$\varphi^* d\text{vol}_g = D(\varphi) \cdot d\text{vol}_g. \quad (47)$$

As a matter of notation, we shall write $D(\varphi) = \varphi^* d\text{vol}_g / d\text{vol}_g$.

It may be shown that the map D is smooth (see [Ham82]), and that

$$D(\varphi \circ \psi) = D(\varphi) \circ \psi \cdot D(\psi), \quad (48)$$

where $\varphi, \psi \in \text{Diff}(M)$.

Observe also that if a diffeomorphism φ preserves the orientation of $(M, d\text{vol}_g)$, then $1/\text{Vol}(M) \cdot D(\varphi) \in \mathcal{D}$.

Lemma 4.2. *The flow φ_t^X of X , is given, for $\rho \in \mathcal{D}$ and $\nabla\phi \in \nabla C^\infty(M)$, by*

$$\varphi_t^X(\rho, \nabla\phi) := \left(\frac{1}{\text{Vol}(M)} \cdot D(\varphi \circ \varphi_t^{\nabla\psi_1}), \nabla\phi + t\nabla\psi_2 \right), \quad (49)$$

where $\varphi \in \text{Diff}(M)$ is chosen such that $D(\varphi) = \text{Vol}(M)\rho$ (such φ necessarily exists according to Moser's Theorem).

Proof. According to (47), we have :

$$\begin{aligned} & D(\varphi \circ \varphi_t^{\nabla\psi_1}) \cdot d\text{vol}_g = (\varphi_t^{\nabla\psi_1})^* \varphi^* d\text{vol}_g = (\varphi_t^{\nabla\psi_1})^* D(\varphi) \cdot d\text{vol}_g \\ \Rightarrow & D(\varphi \circ \varphi_t^{\nabla\psi_1}) \cdot d\text{vol}_g = (\varphi_t^{\nabla\psi_1})^* D(\varphi) \cdot d\text{vol}_g \\ \Rightarrow & \frac{d}{dt} D(\varphi \circ \varphi_t^{\nabla\psi_1}) \cdot d\text{vol}_g = \mathcal{L}_{\nabla\psi_1} \left((\varphi_t^{\nabla\psi_1})^* D(\varphi) \cdot d\text{vol}_g \right), \end{aligned} \quad (50)$$

and, in view of (48),

$$\begin{aligned} & \mathcal{L}_{\nabla\psi_1} \left((\varphi_t^{\nabla\psi_1})^* D(\varphi) \cdot d\text{vol}_g \right) \\ = & \mathcal{L}_{\nabla\psi_1} \left((D(\varphi) \circ \varphi_t^{\nabla\psi_1}) \cdot D(\varphi_t^{\nabla\psi_1}) \cdot d\text{vol}_g \right) \\ = & \mathcal{L}_{\nabla\psi_1} \left(D(\varphi \circ \varphi_t^{\nabla\psi_1}) \cdot d\text{vol}_g \right) \\ = & \left(g(\nabla\psi_1, \nabla D(\varphi \circ \varphi_t^{\nabla\psi_1})) + D(\varphi \circ \varphi_t^{\nabla\psi_1}) \text{div}(\nabla\psi_1) \right) \cdot d\text{vol}_g \\ = & \text{div} \left(D(\varphi \circ \varphi_t^{\nabla\psi_1}) \cdot \nabla\psi_1 \right) \cdot d\text{vol}_g. \end{aligned} \quad (51)$$

Collecting (50) and (51), we thus get

$$\frac{d}{dt} D(\varphi \circ \varphi_t^{\nabla\psi_1}) = \text{div} \left(D(\varphi \circ \varphi_t^{\nabla\psi_1}) \cdot \nabla\psi_1 \right), \quad (52)$$

from which we see, having in mind the identification (36), that

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{\text{Vol}(M)} \cdot D(\varphi \circ \varphi_t^{\nabla\psi_1}), \nabla\phi + t\nabla\psi_2 \right) \\ = & \left(\frac{1}{\text{Vol}(M)} \cdot D(\varphi \circ \varphi_t^{\nabla\psi_1}), \nabla\phi + t\nabla\psi_2, \nabla\psi_1, \nabla\psi_2 \right). \end{aligned} \quad (53)$$

Equation (53) exactly means that φ_t^X , such as defined in (49), is the flow of X . The lemma follows. \square

We are now almost able to compute the Lie bracket $[X, Y]$. But for this, we still need, for $\rho \in \mathcal{D}$, the following continuous map of Fréchet spaces

$$\mathbb{P}_\rho : \begin{cases} \mathfrak{X}(M) = \mathfrak{X}_{d\text{vol}_g}(M) \oplus \rho \nabla C^\infty(M) \rightarrow \nabla C^\infty(M), \\ X = \overline{X} + \rho \nabla \phi \mapsto \nabla \phi, \end{cases} \quad (54)$$

where $\overline{X} \in \mathfrak{X}_{d\text{vol}_g}(M) = \{Z \in \mathfrak{X}(M) \mid \text{div}(Z) = 0\}$, and where the topological direct sum $\mathfrak{X}(M) = \mathfrak{X}_{d\text{vol}_g}(M) \oplus \rho \nabla C^\infty(M)$ is simply a slight generalisation of the Helmholtz-Hodge decomposition (see (24) and [Mol10] for a proof of this generalization).

Remark 4.3. *Using Stokes' Theorem, it is easy to show the following convenient formula :*

$$\int_M g(\nabla \phi, \mathbb{P}_\rho(\rho X)) \rho \cdot d\text{vol}_g = \int_M g(\nabla \phi, X) \rho \cdot d\text{vol}_g, \quad (55)$$

where $\phi \in C^\infty(M)$ and where $X \in \mathfrak{X}(M)$.

Lemma 4.4. *For $\rho \in \mathcal{D}$ and $\nabla \phi \in \nabla C^\infty(M)$, we have :*

$$[X, Y]_{(\rho, \nabla \phi)} = \left(\rho, \nabla \phi, \mathbb{P}_\rho(\rho [\nabla \alpha_1, \nabla \psi_1]), 0 \right). \quad (56)$$

Proof. Let us choose $\varphi, \psi_t, \beta_{t,s} \in \text{Diff}(M)$ such that

$$D(\varphi) = \text{Vol}(M) \cdot \rho, \quad D(\psi_t) = D(\varphi \circ \varphi_t^{\nabla \psi_1}), \quad D(\beta_{t,s}) = D(\psi_t \circ \varphi_s^{\nabla \alpha_1}). \quad (57)$$

According to Lemma 4.2, we have

$$\begin{aligned} [X, Y]_{(\rho, \nabla \phi)} &= \frac{d}{dt} \Big|_0 \left((\varphi_{-t}^X)_* Y_{\varphi_t^X(\rho, \nabla \phi)} \right) \\ &= \frac{d}{dt} \Big|_0 \frac{d}{ds} \Big|_0 \left(\varphi_{-t}^X \circ \varphi_s^Y \circ \varphi_t^X \right) (\rho, \nabla \phi) \\ &= \frac{d}{dt} \Big|_0 \frac{d}{ds} \Big|_0 \left(\varphi_{-t}^X \circ \varphi_s^Y \right) \left(\frac{1}{\text{Vol}(M)} \cdot D(\varphi \circ \varphi_t^{\nabla \psi_1}), \nabla \phi + t \nabla \psi_2 \right) \\ &= \frac{d}{dt} \Big|_0 \frac{d}{ds} \Big|_0 \varphi_{-t}^X \left(\frac{1}{\text{Vol}(M)} \cdot D(\psi_t \circ \varphi_s^{\nabla \alpha_1}), \nabla \phi + t \nabla \psi_2 + s \nabla \alpha_2 \right) \\ &= \frac{d}{dt} \Big|_0 \frac{d}{ds} \Big|_0 \left(\frac{1}{\text{Vol}(M)} \cdot D(\beta_{t,s} \circ \varphi_{-t}^{\nabla \psi_1}), \nabla \phi + t \nabla \psi_2 + s \nabla \alpha_2 - t \nabla \psi_2 \right) \\ &= \frac{d}{dt} \Big|_0 \frac{d}{ds} \Big|_0 \left(\frac{1}{\text{Vol}(M)} \cdot D(\beta_{t,s} \circ \varphi_{-t}^{\nabla \psi_1}), \nabla \phi + s \nabla \alpha_2 \right). \end{aligned} \quad (58)$$

From (58), we already see that the bracket $[X, Y]_{(\rho, \nabla \phi)}$ is of the form $(\rho, \nabla \phi, *, 0)$, where “*” has to be determined by computing the derivatives of $D(\beta_{t,s} \circ \varphi_{-t}^{\nabla \psi_1})$ with respect to s and t , and by putting it in a divergence form.

Using (48) and (57), we see that

$$\begin{aligned}
D(\beta_{t,s} \circ \varphi_{-t}^{\nabla\psi_1}) &= D(\beta_{t,s}) \circ \varphi_{-t}^{\nabla\psi_1} \cdot D(\varphi_{-t}^{\nabla\psi_1}) = D(\psi_t \circ \varphi_s^{\nabla\alpha_1}) \circ \varphi_{-t}^{\nabla\psi_1} \cdot D(\varphi_{-t}^{\nabla\psi_1}) \\
&= D(\psi_t) \circ \varphi_s^{\nabla\alpha_1} \circ \varphi_{-t}^{\nabla\psi_1} \cdot D(\varphi_s^{\nabla\alpha_1}) \circ \varphi_{-t}^{\nabla\psi_1} \cdot D(\varphi_{-t}^{\nabla\psi_1}) \\
&= D(\varphi \circ \varphi_t^{\nabla\psi_1}) \circ \varphi_s^{\nabla\alpha_1} \circ \varphi_{-t}^{\nabla\psi_1} \cdot D(\varphi_s^{\nabla\alpha_1}) \circ \varphi_{-t}^{\nabla\psi_1} \cdot D(\varphi_{-t}^{\nabla\psi_1}) \\
&= \left(D(\varphi \circ \varphi_t^{\nabla\psi_1}) \circ \varphi_s^{\nabla\alpha_1} \cdot D(\varphi_s^{\nabla\alpha_1}) \right) \circ \varphi_{-t}^{\nabla\psi_1} \cdot D(\varphi_{-t}^{\nabla\psi_1}) \\
&= D(\varphi \circ \varphi_t^{\nabla\psi_1} \circ \varphi_s^{\nabla\alpha_1}) \circ \varphi_{-t}^{\nabla\psi_1} \cdot D(\varphi_{-t}^{\nabla\psi_1}) \\
&= D(\varphi \circ \varphi_t^{\nabla\psi_1} \circ \varphi_s^{\nabla\alpha_1} \circ \varphi_{-t}^{\nabla\psi_1}),
\end{aligned} \tag{59}$$

and thus,

$$\begin{aligned}
&\frac{d}{dt} \Big|_0 \frac{d}{ds} \Big|_0 \frac{1}{\text{Vol}(M)} \cdot D(\beta_{t,s} \circ \varphi_{-t}^{\nabla\psi_1}) \\
&= \frac{d}{dt} \Big|_0 \frac{d}{ds} \Big|_0 \frac{1}{\text{Vol}(M)} \cdot D(\varphi \circ \varphi_t^{\nabla\psi_1} \circ \varphi_s^{\nabla\alpha_1} \circ \varphi_{-t}^{\nabla\psi_1}) \\
&= \frac{d}{dt} \Big|_0 \frac{d}{ds} \Big|_0 \rho \circ \varphi_t^{\nabla\psi_1} \circ \varphi_s^{\nabla\alpha_1} \circ \varphi_{-t}^{\nabla\psi_1} \cdot D(\varphi_t^{\nabla\psi_1} \circ \varphi_s^{\nabla\alpha_1} \circ \varphi_{-t}^{\nabla\psi_1}) \\
&= \frac{d}{dt} \Big|_0 \frac{d}{ds} \Big|_0 \rho \circ \varphi_t^{\nabla\psi_1} \circ \varphi_s^{\nabla\alpha_1} \circ \varphi_{-t}^{\nabla\psi_1} + \rho \cdot \frac{d}{dt} \Big|_0 \frac{d}{ds} \Big|_0 D(\varphi_t^{\nabla\psi_1} \circ \varphi_s^{\nabla\alpha_1} \circ \varphi_{-t}^{\nabla\psi_1}) \\
&= g(\nabla\rho, [\nabla\alpha_1, \nabla\psi_1]) + \rho \cdot \frac{d}{dt} \Big|_0 \frac{d}{ds} \Big|_0 (\varphi_t^{\nabla\psi_1} \circ \varphi_s^{\nabla\alpha_1} \circ \varphi_{-t}^{\nabla\psi_1})^* d\text{vol}_g / d\text{vol}_g \\
&= g(\nabla\rho, [\nabla\alpha_1, \nabla\psi_1]) + \rho \cdot \frac{d}{dt} \Big|_0 \mathcal{L}_{(\varphi_t^{\nabla\psi_1})_* \varphi_{-t}^{\nabla\psi_1} (\nabla\alpha_1) \varphi_{-t}^{\nabla\psi_1}} (d\text{vol}_g) / d\text{vol}_g \\
&= g(\nabla\rho, [\nabla\alpha_1, \nabla\psi_1]) + \rho \cdot \frac{d}{dt} \Big|_0 \text{div} \left((\varphi_t^{\nabla\psi_1})_* \varphi_{-t}^{\nabla\psi_1} (\nabla\alpha_1) \varphi_{-t}^{\nabla\psi_1} \right) \\
&= g(\nabla\rho, [\nabla\alpha_1, \nabla\psi_1]) + \rho \cdot \text{div}([\nabla\alpha_1, \nabla\psi_1]) \\
&= \text{div}(\rho \cdot [\nabla\alpha_1, \nabla\psi_1]) = \text{div}(\rho \cdot \mathbb{P}_\rho(\rho \cdot [\nabla\alpha_1, \nabla\psi_1])).
\end{aligned} \tag{60}$$

The lemma follows. \square

Proposition 4.5. *The form $\Omega_{\mathcal{L}} := -d\Theta_{\mathcal{L}}$ (see (39) for the definition of $\Theta_{\mathcal{L}}$), is a symplectic form on $T\mathcal{D}$, and for $\rho \in \mathcal{D}$ and $\nabla\phi \in \nabla C^\infty(M)$,*

$$\begin{aligned}
&(\Omega_{\mathcal{L}})_{(\rho, \nabla\phi)}((\rho, \nabla\phi, \nabla\psi_1, \nabla\psi_2), (\rho, \nabla\phi, \nabla\alpha_1, \nabla\alpha_2)) \\
&= \int_M g(\nabla\psi_1, \nabla\alpha_2) \rho \cdot d\text{vol}_g - \int_M g(\nabla\alpha_1, \nabla\psi_2) \rho \cdot d\text{vol}_g,
\end{aligned} \tag{61}$$

where $\nabla\psi_1, \nabla\psi_2, \nabla\alpha_1, \nabla\alpha_2 \in \nabla C^\infty(M)$.

Proof. The fact that $\Omega_{\mathcal{L}}$ is a symplectic form, i.e., that $\Omega_{\mathcal{L}}$ is non-degenerate (the closedness being clear), is a simple consequence of formula (61) that we are now going to show.

Equation (40), together with Lemma 4.1 and Lemma 4.4, yield

$$\begin{aligned}
& (\Omega_{\mathcal{L}})_{(\rho, \nabla \phi)}((\rho, \nabla \phi, \nabla \psi_1, \nabla \psi_2), (\rho, \nabla \phi, \nabla \alpha_1, \nabla \alpha_2)) \\
&= -(d\Theta_{\mathcal{L}})_{(\rho, \nabla \phi)}(X_{(\rho, \nabla \phi)}, Y_{(\rho, \nabla \phi)}) \\
&= -X_{(\rho, \nabla \phi)}(\Theta_{\mathcal{L}}(Y)) + Y_{(\rho, \nabla \phi)}(\Theta_{\mathcal{L}}(X)) + (\Theta_{\mathcal{L}})_{(\rho, \nabla \phi)}([X, Y]), \\
&= \int_M g(\nabla \psi_1, \nabla \alpha_2) \rho \cdot d\text{vol}_g - \int_M g(\nabla \alpha_1, \nabla \psi_2) \rho \cdot d\text{vol}_g \\
&\quad + \int_M g(\nabla \psi_1, \nabla \phi) \text{div}(\rho \cdot \nabla \alpha_1) \cdot d\text{vol}_g - \int_M g(\nabla \alpha_1, \nabla \phi) \text{div}(\rho \cdot \nabla \psi_1) \cdot d\text{vol}_g \\
&\quad - \int_M g(\nabla \phi, \mathbb{P}_{\rho}(\rho \cdot [\nabla \psi_1, \nabla \alpha_1])) \rho \cdot d\text{vol}_g. \tag{62}
\end{aligned}$$

Clearly, we have to show that the last two lines in (62) vanish.

Using Remark 4.3, one may rewrite the last term in (62) as

$$\int_M g(\nabla \phi, \mathbb{P}_{\rho}(\rho \cdot [\nabla \psi_1, \nabla \alpha_1])) \rho \cdot d\text{vol}_g = \int_M g(\nabla \phi, [\nabla \psi_1, \nabla \alpha_1]) \rho \cdot d\text{vol}_g. \tag{63}$$

Using this last equation, one observes that the last three terms in (62) may be rewritten :

$$\begin{aligned}
& \int_M g(\nabla \psi_1, \nabla \phi) \text{div}(\rho \cdot \nabla \alpha_1) \cdot d\text{vol}_g - \int_M g(\nabla \alpha_1, \nabla \phi) \text{div}(\rho \cdot \nabla \psi_1) \cdot d\text{vol}_g \\
& - \int_M g(\nabla \phi, [\nabla \psi_1, \nabla \alpha_1]) \rho \cdot d\text{vol}_g. \\
&= \int_M \left(-g(\nabla \alpha_1, \nabla g(\nabla \psi_1, \nabla \phi)) + g(\nabla \psi_1, \nabla g(\nabla \alpha_1, \nabla \phi)) - g(\nabla \phi, [\nabla \psi_1, \nabla \alpha_1]) \right) \rho \cdot d\text{vol}_g \\
&= \int_M \left(-(\nabla \alpha_1) d\phi(\nabla \psi_1) + (\nabla \psi_1) d\phi(\nabla \alpha_1) - d\phi([\nabla \psi_1, \nabla \alpha_1]) \right) \rho \cdot d\text{vol}_g \\
&= \int_M d(d\phi)(\nabla \psi_1, \nabla \alpha_1) \rho \cdot d\text{vol}_g = 0. \tag{64}
\end{aligned}$$

The proposition follows. \square

With such simple expression for the symplectic form $\Omega_{\mathcal{L}}$ (see (61)), it is possible to compute explicitly the symplectic gradient of interesting functions, as well as their Poisson brackets. Indeed, we define, for $F : TM \rightarrow \mathbb{R}$, the following function on $T\mathcal{D}$:

$$\widehat{F}(\rho, \nabla \phi) := \int_M F(\nabla \phi) \rho \cdot d\text{vol}_g. \tag{65}$$

We also denote by $\mathcal{H} : T\mathcal{D} \rightarrow \mathbb{R}$, the Hamiltonian associated, via the Legendre transform, to the Lagrangian \mathcal{L} ⁷ :

$$\mathcal{H}(\rho, \nabla \phi) := \int_M \left(\frac{1}{2} \|\nabla \phi\|^2 + V \right) \rho \cdot d\text{vol}_g + \frac{\hbar^2}{2} \int_M \|\nabla(\sqrt{\rho})\|^2 \cdot d\text{vol}_g. \tag{66}$$

⁷Recall that if $L : TM \rightarrow \mathbb{R}$ is a Lagrangian defined on a manifold M , then its associated Hamiltonian $H : TM \rightarrow \mathbb{R}$ is the function defined, for $u_x \in T_x M$, by $H(u_x) := \mathbb{F}L(u_x)(u_x) - L(u_x)$, where $\mathbb{F}L : TM \rightarrow T^*M$ is the Legendre transform of L .

We shall denote by $X_{\widehat{F}}$ and $X_{\mathcal{H}}$ the symplectic gradients associated to \widehat{F} and \mathcal{H} via the symplectic form $\Omega_{\mathcal{L}}$ (recall that these two vector fields are defined on $T\mathcal{D}$ via the relations $\Omega_{\mathcal{L}}(X_{\widehat{F}}, \cdot) = d\widehat{F}$ and $\Omega_{\mathcal{L}}(X_{\mathcal{H}}, \cdot) = d\mathcal{H}$).

On $T\mathcal{D}$, we shall use the Poisson bracket $\{\cdot, \cdot\}_{\mathcal{L}}$ associated to the symplectic form $\Omega_{\mathcal{L}}$ (of course, this Poisson bracket is only defined for functions having a symplectic gradient), and on TM we shall use the Poisson bracket, denoted $\{\cdot, \cdot\}_L$, canonically associated to the Lagrangian $L(u_x) := 1/2 \cdot g(u_x, u_x) - V(x)$.

Proposition 4.6. *For $F, G : TM \rightarrow \mathbb{R}$, $\rho \in \mathcal{D}$ and $\nabla\phi \in \nabla C^\infty(M)$, we have :*

1. $(X_{\mathcal{H}})_{(\rho, \nabla\phi)} = \left(\rho, \nabla\phi, \nabla\phi, \nabla \left[\frac{1}{2} \|\nabla\phi\|^2 + V - \frac{\hbar^2}{2} \frac{\Delta(\sqrt{\rho})}{\sqrt{\rho}} \right] \right),$
2. $(X_{\widehat{F}})_{(\rho, \nabla\phi)} = \left(\rho, \nabla\phi, \mathbb{P}_\rho(\rho(\pi_*^{TM} \circ X_F \circ \nabla\phi)), \nabla(F(\nabla\phi)) \right),$
3. $\{\widehat{F}, \widehat{G}\}_{\mathcal{L}} = -\widehat{\{F, G\}_L}.$

We will show Proposition 4.6 with a series of Lemmas.

Lemma 4.7. *For $\rho \in \mathcal{D}$ and $\nabla\phi \in \nabla C^\infty(M)$, we have :*

$$(X_{\mathcal{H}})_{(\rho, \nabla\phi)} = \left(\rho, \nabla\phi, \nabla\phi, \nabla \left[\frac{1}{2} \|\nabla\phi\|^2 + V - \frac{\hbar^2}{2} \frac{\Delta(\sqrt{\rho})}{\sqrt{\rho}} \right] \right). \quad (67)$$

Proof. We will use the vector field $X \in \mathfrak{X}(T\mathcal{D})$ introduced in (41), and especially its flow φ_t^X which is given in Lemma 4.2.

We have :

$$\begin{aligned} (d\mathcal{H})_{(\rho, \nabla\phi)} X_{(\rho, \nabla\phi)} &= \frac{d}{dt} \Big|_0 (\mathcal{H} \circ \varphi_t^X)(\rho, \nabla\phi) \\ &= \frac{d}{dt} \Big|_0 \mathcal{H} \left(\frac{1}{\text{Vol}(M)} \cdot D(\varphi \circ \varphi_t^{\nabla\psi_1}), \nabla\phi + t\nabla\psi_2 \right) \\ &= \frac{d}{dt} \Big|_0 \left[\int_M \left(\frac{1}{2} \|\nabla\phi + t\nabla\psi_2\|^2 + V \right) \frac{1}{\text{Vol}(M)} D(\varphi \circ \varphi_t^{\nabla\psi_1}) \cdot d\text{vol}_g \right. \\ &\quad \left. + \frac{\hbar^2}{2} \int_M \left\| \nabla \left(\sqrt{\frac{1}{\text{Vol}(M)}} D(\varphi \circ \varphi_t^{\nabla\psi_1}) \right) \right\|^2 \cdot d\text{vol}_g \right] \\ &= \frac{d}{dt} \Big|_0 \int_M \left(\frac{1}{2} \|\nabla\phi\|^2 + t g(\nabla\phi, \nabla\psi_2) + \frac{t^2}{2} \|\nabla\psi_2\|^2 + V \right) \rho_t \cdot d\text{vol}_g \\ &\quad + \frac{d}{dt} \Big|_0 \frac{\hbar^2}{2} \int_M \|\nabla(\sqrt{\rho_t})\|^2 \cdot d\text{vol}_g, \end{aligned} \quad (68)$$

where $\rho_t := 1/\text{Vol}(M) \cdot D(\varphi \circ \varphi_t^{\nabla\psi_1})$.

But, according to (52),

$$\frac{\partial \rho_t}{\partial t} = \text{div}(\rho_t \cdot \nabla\psi_1), \quad (69)$$

and thus,

$$\begin{aligned}
& (d\mathcal{H})_{(\rho, \nabla \phi)} X_{(\rho, \nabla \phi)} \\
&= \frac{1}{2} \int_M \|\nabla \phi\|^2 \operatorname{div}(\rho \cdot \nabla \psi_1) \cdot d\operatorname{vol}_g + \int_M g(\nabla \phi, \nabla \psi_2) \rho \cdot d\operatorname{vol}_g \\
&+ \int_M V \operatorname{div}(\rho \cdot \nabla \psi_1) \cdot d\operatorname{vol}_g + \frac{d}{dt} \Big|_0 \frac{\hbar^2}{2} \int_M \|\nabla(\sqrt{\rho_t})\|^2 \cdot d\operatorname{vol}_g.
\end{aligned} \tag{70}$$

Let us compute the last term in (70) :

$$\begin{aligned}
& \frac{d}{dt} \Big|_0 \int_M \|\nabla(\sqrt{\rho_t})\|^2 \cdot d\operatorname{vol}_g = 2 \int_M g\left(\nabla \frac{\partial}{\partial t} \Big|_0 \sqrt{\rho_t}, \nabla(\sqrt{\rho})\right) \cdot d\operatorname{vol}_g \\
&= \int_M g\left(\nabla \left[\frac{1}{\sqrt{\rho}} \operatorname{div}(\rho \cdot \nabla \psi_1)\right], \nabla(\sqrt{\rho})\right) \cdot d\operatorname{vol}_g \\
&= \int_M g\left(\operatorname{div}(\rho \cdot \nabla \psi_1) \left(-\frac{1}{\rho} \cdot \frac{1}{2\sqrt{\rho}} \nabla \rho\right) + \frac{1}{\sqrt{\rho}} \nabla \operatorname{div}(\rho \cdot \nabla \psi_1), \nabla(\sqrt{\rho})\right) \cdot d\operatorname{vol}_g \\
&= - \int_M g(\nabla \rho, \nabla(\sqrt{\rho})) \cdot \operatorname{div}(\rho \cdot \nabla \psi_1) \frac{1}{\rho} \cdot \frac{1}{2\sqrt{\rho}} \cdot d\operatorname{vol}_g + \int_M g(\nabla \operatorname{div}(\rho \cdot \nabla \psi_1), \nabla(\sqrt{\rho})) \frac{1}{\sqrt{\rho}} \cdot d\operatorname{vol}_g \\
&= - \int_M \|\nabla \rho\|^2 \cdot \operatorname{div}(\rho \cdot \nabla \psi_1) \frac{1}{4} \cdot \frac{1}{\rho^2} \cdot d\operatorname{vol}_g + \int_M g(\nabla \operatorname{div}(\rho \cdot \nabla \psi_1), \nabla \rho) \frac{1}{2\rho} \cdot d\operatorname{vol}_g \\
&= - \int_M \|\nabla \rho\|^2 \cdot \operatorname{div}(\rho \cdot \nabla \psi_1) \frac{1}{4} \cdot \frac{1}{\rho^2} \cdot d\operatorname{vol}_g - \int_M \operatorname{div}(\rho \cdot \nabla \psi_1) \cdot \mathcal{L}_{\nabla \rho} \left(\frac{1}{2\rho}\right) \cdot d\operatorname{vol}_g \\
&= - \int_M \|\nabla \rho\|^2 \cdot \operatorname{div}(\rho \cdot \nabla \psi_1) \frac{1}{4} \cdot \frac{1}{\rho^2} \cdot d\operatorname{vol}_g - \int_M \operatorname{div}(\rho \cdot \nabla \psi) g(\nabla \rho, \nabla(\frac{1}{2\rho})) \cdot d\operatorname{vol}_g \\
&\quad - \int_M \operatorname{div}(\rho \cdot \nabla \psi_1) \frac{1}{2\rho} \cdot \operatorname{div}(\nabla \rho) \cdot d\operatorname{vol}_g \\
&= \int_M \left[\frac{1}{4} \frac{\|\nabla \rho\|^2}{\rho^2} - \frac{1}{2} \frac{\Delta \rho}{\rho} \right] \operatorname{div}(\rho \cdot \nabla \psi_1) \cdot d\operatorname{vol}_g = - \int_M \frac{\Delta(\sqrt{\rho})}{\sqrt{\rho}} \operatorname{div}(\rho \cdot \nabla \psi_1) \cdot d\operatorname{vol}_g.
\end{aligned} \tag{71}$$

In the above computation, we have used the following formula,

$$\frac{1}{4} \frac{\|\nabla u\|^2}{u^2} - \frac{1}{2} \frac{\Delta u}{u} = - \frac{\Delta(\sqrt{u})}{\sqrt{u}}, \tag{72}$$

which is valid for every smooth function $u : M \rightarrow \mathbb{R}$, as one may see after a little computation.

Now, (70), (71) and Proposition 4.5 yield

$$\begin{aligned}
& (d\mathcal{H})_{(\rho, \nabla \phi)} X_{(\rho, \nabla \phi)} = \\
& \int_M \left[\frac{1}{2} \|\nabla \phi\|^2 + V - \frac{\hbar^2}{2} \frac{\Delta(\sqrt{\rho})}{\sqrt{\rho}} \right] \operatorname{div}(\rho \cdot \nabla \psi_1) \cdot d\operatorname{vol}_g + \int_M g(\nabla \phi, \nabla \psi_2) \rho \cdot d\operatorname{vol}_g \\
&= - \int_M g\left(\nabla \psi_1, \nabla \left[\frac{1}{2} \|\nabla \phi\|^2 + V - \frac{\hbar^2}{2} \frac{\Delta(\sqrt{\rho})}{\sqrt{\rho}} \right]\right) \rho \cdot d\operatorname{vol}_g + \int_M g(\nabla \phi, \nabla \psi_2) \rho \cdot d\operatorname{vol}_g \\
&= (\Omega_{\mathcal{L}})_{(\rho, \nabla \phi)}(X_{\mathcal{H}}, X).
\end{aligned} \tag{73}$$

The lemma follows. \square

Remark 4.8. We observe (as it was intended to), that the flow generated by the symplectic gradient $X_{\mathcal{H}} \in \mathfrak{X}(T\mathcal{D})$ corresponds exactly to the solutions of the Euler-Lagrange equations on \mathcal{D} associated to the Lagrangian $\mathcal{L} : T\mathcal{D} \rightarrow \mathbb{R}$ introduced in (33), i.e., it satisfies the system of equations (27) (with $X \cong 0$). We thus have a rigorous symplectic formulation of the Schrödinger equation via its hydrodynamical formulation which agrees with the corresponding Lagrangian formulation given in Corollary 3.5.

Lemma 4.9. For $\rho \in \mathcal{D}$, $\nabla\phi \in \nabla C^\infty(M)$ and $F : TM \rightarrow \mathbb{R}$, we have :

$$(X_{\widehat{F}})_{(\rho, \nabla\phi)} = \left(\rho, \nabla\phi, \mathbb{P}_\rho(\rho(\pi_*^{TM} \circ X_F \circ \nabla\phi), \nabla(F(\nabla\phi))) \right). \quad (74)$$

Proof. As for the proof of Lemma 4.7, we will use the vector field $X \in \mathfrak{X}(T\mathcal{D})$ introduced in (41), its flow φ_t^X which is given in Lemma 4.2, and the curve ρ_t defined in the proof of Lemma 4.7 (see (69)). We have :

$$\begin{aligned} (d\widehat{F})_{(\rho, \nabla\phi)} X_{(\rho, \nabla\phi)} &= \frac{d}{dt} \Big|_0 \widehat{F}(\rho_t, \nabla\phi + t\nabla\psi_2) \\ &= \frac{d}{dt} \Big|_0 \int_M F(\nabla\phi + t\nabla\psi_2) \rho_t \cdot d\text{vol}_g \\ &= \int_M \left[\mathbb{F}F(\nabla\phi)(\nabla\psi_2) \rho + F(\nabla\phi) \text{div}(\rho \cdot \nabla\phi) \right] \cdot d\text{vol}_g \\ &= \int_M \left[\mathbb{F}F(\nabla\phi)(\nabla\psi_2) - g(\nabla\psi_1, \nabla(F(\nabla\phi))) \right] \rho \cdot d\text{vol}_g. \end{aligned} \quad (75)$$

We need to transform the term $\mathbb{F}F(\nabla\phi)(\nabla\psi_2)$ into a scalar product; to this end, we will use the following formula

$$\mathbb{F}F(u_x)(v_x) = g_x(\pi_{*u_x}^{TM}(X_F)_{u_x}, v_x), \quad (76)$$

which holds whenever $u_x, v_x \in T_x M$, and where X_F is the symplectic gradient of F with respect to the symplectic form ω on TM canonically associated to the metric g . This formula may be seen as follows. Recall that the canonical symplectic form ω may be written (see [Lan02] and Example 5.1) :

$$\omega_{u_x}(A_{u_x}, B_{u_x}) = g_x(\pi_{*u_x}^{TM} A_{u_x}, K B_{u_x}) - g_x(\pi_{*u_x}^{TM} B_{u_x}, K A_{u_x}), \quad (77)$$

where $u_x \in T_x M$, $A_{u_x}, B_{u_x} \in T_{u_x} TM$ and where $K : T(TM) \rightarrow TM$ is the connector associated to the Riemannian metric g . With (77), it is a simple matter to derive (76) :

$$\begin{aligned} \mathbb{F}F(u_x)(v_x) &= \frac{d}{dt} \Big|_0 F(u_x + tv_x) = (dF)_{u_x} \frac{d}{dt} \Big|_0 (u_x + tv_x) = \omega_{u_x} \left((X_F)_{u_x}, \frac{d}{dt} \Big|_0 (u_x + tv_x) \right) \\ &= g_x \left(\pi_{*u_x}^{TM}(X_F)_{u_x}, K \frac{d}{dt} \Big|_0 (u_x + tv_x) \right) - g_x \left(\pi_{*u_x}^{TM} \frac{d}{dt} \Big|_0 (u_x + tv_x), K(X_F)_{u_x} \right) \\ &= g_x(\pi_{*u_x}^{TM}(X_F)_{u_x}, v_x). \end{aligned} \quad (78)$$

Of course, in the above computation we have used the following simple formulas:

$$K \frac{d}{dt} \Big|_0 (u_x + tv_x) = v_x \quad \text{and} \quad \pi_{*u_x}^{TM} \frac{d}{dt} \Big|_0 (u_x + tv_x) = 0. \quad (79)$$

Taking into account (76), we may rewrite (75) as

$$\begin{aligned}
& \int_M \left[\mathbb{F}F(\nabla\phi)(\nabla\psi_2) - g(\nabla\psi_1, \nabla(F(\nabla\phi))) \right] \rho \cdot d\text{vol}_g \\
&= \int_M \left[g(\pi_*^{TM} \circ X_F \circ \nabla\phi, \nabla\psi_2) - g(\nabla\psi_1, \nabla(F(\nabla\phi))) \right] \rho \cdot d\text{vol}_g \\
&= \int_M \left[g(\mathbb{P}_\rho(\rho(\pi_*^{TM} \circ X_F \circ \nabla\phi)), \nabla\psi_2) - g(\nabla\psi_1, \nabla(F(\nabla\phi))) \right] \rho \cdot d\text{vol}_g, \tag{80}
\end{aligned}$$

from which we see that $(d\widehat{F})X = \Omega_{\mathcal{L}}(X_{\widehat{F}}, X)$, with $X_{\widehat{F}}$ such as defined in the right hand side of (74). The vector field $X_{\widehat{F}}$ is thus the symplectic gradient of F with respect to the symplectic form $\Omega_{\mathcal{L}}$. The lemma follows. \square

Lemma 4.10. *For $F, G : TM \rightarrow \mathbb{R}$, we have :*

$$\{\widehat{F}, \widehat{G}\}_{\mathcal{L}} = -\{\widehat{F}, \widehat{G}\}_L. \tag{81}$$

Proof. For $\rho \in \mathcal{D}$, $\nabla\phi \in \nabla C^\infty(M)$, and, in view of Lemma 4.9, we have :

$$\begin{aligned}
& \{\widehat{F}, \widehat{G}\}_{\mathcal{L}}(\rho, \nabla\phi) = (\Omega_{\mathcal{L}})_{(\rho, \nabla\phi)}(X_{\widehat{F}}, X_{\widehat{G}}) \\
&= \int_M g\left(\mathbb{P}_\rho(\rho(\pi_*^{TM} \circ X_F \circ \nabla\phi)), \nabla(G(\nabla\phi))\right) \rho \cdot d\text{vol}_g \\
&\quad - \int_M g\left(\mathbb{P}_\rho(\rho(\pi_*^{TM} \circ X_G \circ \nabla\phi)), \nabla(F(\nabla\phi))\right) \rho \cdot d\text{vol}_g \\
&= \int_M g\left(\pi_*^{TM} \circ X_F \circ \nabla\phi, \nabla(G(\nabla\phi))\right) \rho \cdot d\text{vol}_g \\
&\quad - \int_M g\left(\pi_*^{TM} \circ X_G \circ \nabla\phi, \nabla(F(\nabla\phi))\right) \rho \cdot d\text{vol}_g. \tag{82}
\end{aligned}$$

Moreover, we observe that if X is a vector field on M , then

$$\begin{aligned}
& g(\nabla(G(\nabla\phi)), X) = G_*(\nabla\phi)_*X = \omega(X_G \circ \nabla\phi, (\nabla\phi)_*X) \\
&= g(\pi_*^{TM} \circ X_G \circ \nabla\phi, K(\nabla\phi)_*X) - g(\pi_*^{TM} \circ (\nabla\phi)_*X, KX_G \circ \nabla\phi) \\
&= g(\pi_*^{TM} \circ X_G \circ \nabla\phi, \nabla_X \nabla\phi) - g(X, KX_G \circ \nabla\phi), \tag{83}
\end{aligned}$$

and thus, denoting $\widetilde{X}_F := \pi_*^{TM} \circ X_G \circ \nabla\phi$ and $\widetilde{X}_G := \pi_*^{TM} \circ X_F \circ \nabla\phi$ for simplicity, we may rewrite (82)

as :

$$\begin{aligned}
& \{\widehat{F}, \widehat{G}\}_{\mathcal{L}}(\rho, \nabla\phi) = \\
& \int_M g(\tilde{X}_G, \nabla_{\tilde{X}_F} \nabla\phi) \rho \cdot d\text{vol}_g - \int_M g(\tilde{X}_F, KX_G \circ \nabla\phi) \rho \cdot d\text{vol}_g \\
& - \int_M g(\tilde{X}_F, \nabla_{\tilde{X}_G} \nabla\phi) \rho \cdot d\text{vol}_g - \int_M g(\tilde{X}_G, KX_F \circ \nabla\phi) \rho \cdot d\text{vol}_g \\
& = - \left[\int_M g(\tilde{X}_F, KX_G \circ \nabla\phi) \rho \cdot d\text{vol}_g - \int_M g(\tilde{X}_G, KX_G \circ \nabla\phi) \rho \cdot d\text{vol}_g \right] \\
& + \int_M g(\tilde{X}_G, \nabla_{\tilde{X}_F} \nabla\phi) \rho \cdot d\text{vol}_g - \int_M g(\tilde{X}_F, \nabla_{\tilde{X}_G} \nabla\phi) \rho \cdot d\text{vol}_g \\
& = -\{\widehat{F}, \widehat{G}\}_L(\rho, \nabla\phi) + \int_M g(\tilde{X}_G, \nabla_{\tilde{X}_F} \nabla\phi) \rho \cdot d\text{vol}_g - \int_M g(\tilde{X}_F, \nabla_{\tilde{X}_G} \nabla\phi) \rho \cdot d\text{vol}_g. \quad (84)
\end{aligned}$$

Clearly, we have to show that the last line in (84) vanishes. But this can be done easily with the help of the following formula

$$g(X, \nabla_Y Z) - g(Y, \nabla_X Z) = -d(Z^\sharp)(X, Y), \quad (85)$$

which holds for every vector fields $X, Y, Z \in \mathfrak{X}(M)$, and where Z^\sharp is the 1-form on M defined by $(Z^\sharp)_x(u_x) := g_x(Z_x, u_x)$, $u_x \in T_x M$.

Using (85) and the fact that $d(d\phi) = 0$, one easily sees that the last line in (84) vanishes. The lemma follows. \square

5 The almost Hermitian structure of $T\mathcal{D}$

In §3 and §4, we used the usual techniques of geometric mechanics to find a Lagrangian and Hamiltonian description of the Schrödinger equation, and we eventually arrived at the symplectic form $\Omega_{\mathcal{L}}$ on $T\mathcal{D}$ which encodes the dynamics of a quantum particle and whose explicit description is given in Proposition 4.5.

In this section, we follow some ideas of [Mol] and show that $\Omega_{\mathcal{L}}$ is the fundamental 2-form of an almost Hermitian structure on $T\mathcal{D}$ which comes from Dombrowski's construction [Dom62] applied to a metric $g^{\mathcal{D}}$ and a (non-metric) connection $\nabla^{\mathcal{D}}$ on \mathcal{D} , and discuss the integrability of this almost Hermitian structure.

Let us start by recalling Dombrowski's construction. If M is a manifold endowed with an affine connection ∇ , then Dombrowski splitting Theorem holds (see [Dom62, Lan02]) :

$$T(TM) \cong TM \oplus TM \oplus TM, \quad (86)$$

this splitting being viewed as an isomorphism of vector bundles over M , and the isomorphism, say Φ , being

$$T_{u_x} TM \ni A_{u_x} \xrightarrow{\Phi} (u_x, (\pi^M)_{*u_x} A_{u_x}, K^M A_{u_x}), \quad (87)$$

where $\pi^M : TM \rightarrow M$ is the canonical projection and where $K^M : T(TM) \rightarrow TM$ is the canonical connector associated to the connection ∇ (see [Lan02]).

Having $A_{u_x} = \Phi^{-1}((u_x, v_x, w_x)) \in T_{u_x}TM$, we shall write, for simplicity, $A_{u_x} = (u_x, v_x, w_x)$ instead of $\Phi^{-1}((u_x, v_x, w_x))$, i.e., we will drop Φ . The second component v_x is usually referred to as the horizontal component of A_{u_x} (with respect to the connection ∇) and w_x the vertical component.

With the above notation, and provided that M is endowed with a Riemannian metric g , it is a simple matter to define on TM an almost Hermitian structure. Indeed, we define a metric g^{TM} , a 2-form ω^{TM} and an almost complex structure J^{TM} by setting

$$\begin{aligned} g_{u_x}^{TM}((u_x, v_x, w_x), (u_x, \bar{v}_x, \bar{w}_x)) &:= g_x(v_x, \bar{v}_x) + g_x(w_x, \bar{w}_x), \\ \omega_{u_x}^{TM}((u_x, v_x, w_x), (u_x, \bar{v}_x, \bar{w}_x)) &:= g_x(v_x, \bar{w}_x) - g_x(w_x, \bar{v}_x), \\ J_{u_x}^{TM}((u_x, v_x, w_x)) &:= (u_x, -w_x, v_x), \end{aligned} \quad (88)$$

where $u_x, v_x, w_x, \bar{v}_x, \bar{w}_x \in T_xM$.

Clearly, $(J^{TM})^2 = -\text{Id}$ and $g^{TM}(J^{TM} \cdot, J^{TM} \cdot) = g^{TM}(\cdot, \cdot)$, which means that (TM, g^{TM}, J^{TM}) is an almost Hermitian manifold, and one readily sees that g^{TM}, J^{TM} and ω^{TM} are compatible, i.e., that $\omega^{TM} = g^{TM}(J^{TM} \cdot, \cdot)$; the 2-form ω^{TM} is thus the fundamental 2-form of the almost Hermitian manifold (TM, g^{TM}, J^{TM}) . This is Dombrowski's construction.

Example 5.1. Let (M, g) be a (finite dimensional) Riemannian manifold with Levi-Civita connection ∇ , and let $\omega = -d\theta$ be the canonical symplectic form⁸ on T^*M . Then the 2-form ω^{TM} on TM associated to (g, ∇) via Dombrowski's construction is equal to the pull back of the canonical symplectic form ω via the Legendre transform $TM \rightarrow T^*M, v_x \mapsto g_x(v_x, \cdot)$ (see [Lan02]).

In the case of the infinite dimensional manifold \mathcal{D} , we already defined in (34) a metric $g^{\mathcal{D}}$ on \mathcal{D} :

$$(g^{\mathcal{D}})_{\rho}((\rho, \nabla\phi), (\rho, \nabla\phi')) := \int_M g(\nabla\phi, \nabla\phi') \rho \cdot d\text{vol}_g, \quad (89)$$

where $\rho \in \mathcal{D}$ and where $\nabla\phi, \nabla\phi' \in \nabla C^{\infty}(M)$. We also used the following identification (see (36)):

$$T(T\mathcal{D}) \cong \mathcal{D} \times \nabla C^{\infty}(M) \times \nabla C^{\infty}(M) \times \nabla C^{\infty}(M). \quad (90)$$

Clearly, this identification defines an affine connection $\nabla^{\mathcal{D}}$ on \mathcal{D} whose associated connector $K^{\mathcal{D}}$ is

$$K^{\mathcal{D}} : T(T\mathcal{D}) \rightarrow T\mathcal{D}, \quad (\rho, \nabla\phi, \nabla\psi_1, \nabla\psi_2) \mapsto (\rho, \nabla\psi_2) \quad (91)$$

(one easily verifies that the above map has the properties of a connector).

We thus have a triple $(\mathcal{D}, g^{\mathcal{D}}, \nabla^{\mathcal{D}})$ which yields, via Dombrowski's construction, an almost Hermitian structure $(g^{T\mathcal{D}}, J^{T\mathcal{D}}, \omega^{T\mathcal{D}})$ on $T\mathcal{D}$. For example,

$$\begin{aligned} &(g^{T\mathcal{D}})_{(\rho, \nabla\phi)}((\rho, \nabla\phi, \nabla\psi_1, \nabla\psi_2), (\rho, \nabla\phi, \nabla\alpha_1, \nabla\alpha_2)) \\ &= \int_M g(\nabla\psi_1, \nabla\alpha_1) \rho \cdot d\text{vol}_g + \int_M g(\nabla\psi_2, \nabla\alpha_2) \rho \cdot d\text{vol}_g. \end{aligned} \quad (92)$$

In particular, Proposition 4.5 immediately yields

⁸Recall that the canonical 1-form θ on T^*M is defined, for $\alpha_x \in T_x^*M$ and $A_{\alpha_x} \in T_{\alpha_x}T^*M$, by $\theta_{\alpha_x}(A_{\alpha_x}) := \alpha_x((\pi^{T^*M})_{*_{\alpha_x}} A_{\alpha_x})$, where $\pi^{T^*M} : T^*M \rightarrow M$ is the canonical projection.

Proposition 5.2. *The fundamental 2-form $\omega^{T\mathcal{D}}$ of the almost Hermitian structure of $T\mathcal{D}$ associated to $(g^{\mathcal{D}}, \nabla^{\mathcal{D}})$ via Dombrowski's construction is $\Omega_{\mathcal{L}}$, i.e.*

$$\omega^{T\mathcal{D}} = \Omega_{\mathcal{L}}, \quad (93)$$

where $\Omega_{\mathcal{L}} = -d\Theta_{\mathcal{L}}$ has been defined in (35).

Remark 5.3. *As we saw in §4, the flow generated by the Hamiltonian vector field $X_{\mathcal{H}} \in \mathfrak{X}(T\mathcal{D})$ with respect to the symplectic form $\Omega_{\mathcal{L}}$ gives the dynamics of a quantum particle under the influence of a potential V (see (66) for the definition of $\mathcal{H} : T\mathcal{D} \rightarrow \mathbb{R}$). Hence, and since $\Omega_{\mathcal{L}} = \Omega^{T\mathcal{D}}$, we deduce that the dynamics of a quantum particle is encoded in $(\mathcal{D}, g^{\mathcal{D}}, \nabla^{\mathcal{D}})$. This is analogous to the fact that the dynamics of a finite dimensional quantum system is encoded in the triple $(\mathcal{P}_n^{\times}, h_F, \nabla^{(e)})$, where h_F and $\nabla^{(e)}$ are respectively the Fisher metric and the exponential connection on \mathcal{P}_n^{\times} (see [Mol, Mol12]). In this sense, $g^{\mathcal{D}}$ and $\nabla^{\mathcal{D}}$ are infinite dimensional analogues of h_F and $\nabla^{(e)}$.*

Let $T^{\mathcal{D}}$ and $R^{\mathcal{D}}$ be the torsion and the curvature tensor associated to the connection $\nabla^{\mathcal{D}}$, i.e.,

- $T^{\mathcal{D}}(X, Y) = \nabla_X^{\mathcal{D}} Y - \nabla_Y^{\mathcal{D}} X - [X, Y],$
- $R^{\mathcal{D}}(X, Y)(Z) = \nabla_X^{\mathcal{D}} \nabla_Y^{\mathcal{D}} Z - \nabla_Y^{\mathcal{D}} \nabla_X^{\mathcal{D}} Z - \nabla_{[X, Y]}^{\mathcal{D}} Z,$

where $X, Y, Z \in \mathfrak{X}(\mathcal{D})$.

By inspection of the proof of Lemma 4.4, one easily finds that

Lemma 5.4. *We have:*

1. $T^{\mathcal{D}}((\rho, \nabla\phi), (\rho, \nabla\psi)) = (\rho, \mathbb{P}_{\rho}(\rho[\nabla\phi, \nabla\psi])),$
2. $R^{\mathcal{D}} \equiv 0,$

where $\rho \in \mathcal{D}$ and $\nabla\phi, \nabla\psi \in \nabla C^{\infty}(M)$, and where the operator \mathbb{P}_{ρ} has been defined in (54). In particular, $\nabla^{\mathcal{D}}$ is not the Levi-Civita connection associated to $g^{\mathcal{D}}$ (its torsion is not trivial).

Let $N^{T\mathcal{D}}$ be the Nijenhuis tensor of $J^{T\mathcal{D}}$, i.e.,

$$N^{T\mathcal{D}}(X, Y) := [X, Y] - [J^{TM} X, J^{TM} Y] + J^{T\mathcal{D}}[J^{T\mathcal{D}} X, Y] + J^{T\mathcal{D}}[X, J^{T\mathcal{D}} Y], \quad (94)$$

where $X, Y \in \mathfrak{X}(T\mathcal{D})$.

Again, by inspection of the proof of Lemma 4.4, one easily finds that

Proposition 5.5. *Let $J^{T\mathcal{D}}$ be the almost complex structure on $T\mathcal{D}$ associated to $(g^{\mathcal{D}}, \nabla^{\mathcal{D}})$ via Dombrowski's construction, and let $N^{T\mathcal{D}}$ be its Nijenhuis tensor. Then,*

$$N^{T\mathcal{D}}((\rho, \nabla\phi, \nabla\psi_1, \nabla\psi_2), (\rho, \nabla\phi, \nabla\alpha_1, \nabla\alpha_2)) = (\rho, \nabla\phi, \mathbb{P}_{\rho}\{\rho[\nabla\alpha_1, \nabla\psi_1] - \rho[\nabla\alpha_2, \nabla\psi_2]\}, 0) + (\rho, \nabla\phi, 0, \mathbb{P}_{\rho}\{\rho[\nabla\psi_2, \nabla\alpha_1] + \rho[\nabla\psi_1, \nabla\alpha_2]\}), \quad (95)$$

where $\rho \in \mathcal{D}$ and where $\nabla\phi, \nabla\psi_1, \nabla\psi_2, \nabla\alpha_1, \nabla\alpha_2 \in \nabla C^{\infty}(M)$.

Corollary 5.6. *The almost Hermitian structure $J^{T\mathcal{D}}$ of $T\mathcal{D}$ is not integrable, i.e., $N^{T\mathcal{D}} \neq 0$.*

6 Discussion : the wave function of a statistical manifold

In §3, we associated to a time-dependant probability density function ρ on a Riemannian manifold (M, g) a “wave function” $\psi := \sqrt{\rho} e^{-\frac{i}{\hbar} \phi}$ whose phase ϕ is determined by solving the partial differential equation $\dot{\rho} = \operatorname{div}(\rho \nabla \phi)$. As we saw, this wave function linearizes the system of equations given in Proposition 3.1 and yields the usual Schrödinger equation.

In this section, *which is mainly heuristic*, we discuss further the correspondence $\dot{\rho} \rightarrow \psi$ through an example⁹, and make several comments and observations which relate ψ to representation theory, Kähler geometry, the geometrical formulation of quantum mechanics and quantization.

Let us start with a simple example. Let $\mathcal{N}(\mu, 1)$ be the space of probability density functions $p(\xi; \mu)$ defined over \mathbb{R} by

$$p(\xi; \mu) := \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{(\mu - \xi)^2}{2} \right\}, \quad (96)$$

where $\xi, \mu \in \mathbb{R}$.

The set $\mathcal{N}(\mu, 1)$ is a 1-dimensional statistical manifold parameterized by the mean $\mu \in \mathbb{R}$, i.e. $\mathcal{N}(\mu, 1) \cong \mathbb{R}$. As one may easily show (see [AN00]), the Fisher metric $h_F(\mu)$ is the Euclidean metric, and the exponential connection $\nabla^{(e)}$ and the mixture connection $\nabla^{(m)}$ are equal to the canonical flat connection. Consequently (see [Mol]), $T\mathcal{N}(\mu, 1)$ is naturally a Kähler manifold (via Dombrowski’s construction) and one sees that $T\mathcal{N}(\mu, 1) \cong \mathbb{C}$ via the map $b \partial_\mu|_a \mapsto a + i b$.

Now, one of the most important ingredients of the geometrical formulation of quantum mechanics is the notion of *Kähler functions*. By definition, a smooth function $f : N \rightarrow \mathbb{R}$ on a Kähler manifold N with Kähler structure (g, J, ω) is a Kähler function if it satisfies $\mathcal{L}_{X_f} g = 0$, where X_f is the Hamiltonian vector field associated to f , i.e. $\omega(X_f, \cdot) = df(\cdot)$, and where \mathcal{L}_{X_f} is the Lie derivative in the direction X_f .

The space of Kähler functions $\mathcal{K}(N)$ on a Kähler manifold is always a finite dimensional Lie algebra for the natural Poisson bracket $\{f, g\} := \omega(X_f, X_g)$. For example, when $N = \mathbb{P}(\mathbb{C}^n)$ is the complex projective space, then $\mathcal{K}(\mathbb{P}(\mathbb{C}^n))$ is isomorphic (in the Lie algebra sense) to the space of $n \times n$ skew Hermitian matrices. Hence, Kähler functions are the natural geometric analogues of the usual observables in quantum mechanics (see [AS99]).

In the case $N = \mathbb{C}$ ($\cong T\mathcal{N}(\mu, 1)$), it is not difficult to see that the space $\mathcal{K}(\mathbb{C})$ of Kähler functions on \mathbb{C} is spanned by

$$1, \ x, \ y, \ \frac{x^2 + y^2}{2} \quad (97)$$

(here x and y are respectively the real and imaginary parts of $z \in \mathbb{C}$), with the following commutators

$$\{1, \cdot\} = 0, \ \{x, y\} = 1, \ \left\{x, \frac{x^2 + y^2}{2}\right\} = y, \ \left\{y, \frac{x^2 + y^2}{2}\right\} = -x. \quad (98)$$

The Lie algebra $\mathcal{K}(\mathbb{C})$ is related to quantum physics. If $p(t)$ is a smooth curve in $\mathcal{N}(\mu, 1)$, it is in particular a smooth curve in $\mathcal{D}(\mathbb{R})$, the space of smooth density probability functions¹⁰ defined over \mathbb{R}

⁹This example has already been discussed in [Mol], but without any mathematical justifications.

¹⁰Even though \mathbb{R} is not compact, the space $\mathcal{D}(\mathbb{R})$ can be given the structure of an infinite dimensional manifold, for example by using the convenient setting developed in [KM97].

for the Lebesgue measure. Moreover, if the time-derivative $\dot{p}(t)$ of $p(t)$ is identified with $x(t) + iy(t) \in \mathbb{C}$, then a direct computation shows that

$$\frac{dp(t)}{dt} = \operatorname{div} (p(t) \nabla \phi), \quad (99)$$

where the (time-dependant) function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is defined (up to an additive constant) by

$$\phi(\xi) = y(t) \xi. \quad (100)$$

Hence, and taking into account (29), the derivative $\dot{p}(t)$ has an associated wave function $\Psi : \mathbb{C} \rightarrow L^2(\mathbb{R}, \mathbb{C})$ which is defined, for $\xi \in \mathbb{R}$ and $z = x + iy \in \mathbb{C}$, by $\Psi(z)(\xi) := \sqrt{p(t)} e^{-\frac{i}{\hbar} \phi(\xi)}$, i.e.,

$$\Psi(z)(\xi) := \frac{1}{(2\pi)^{1/4}} \exp \left\{ -\frac{(\xi - x)^2}{4} \right\} \exp \left\{ -\frac{i}{\hbar} y \xi \right\}. \quad (101)$$

By construction, if $z = x + iy$, then

$$|\Psi(z)(\xi)|^2 = p(\xi; x). \quad (102)$$

The map Ψ is related to quantization and the geometrical formulation of quantum mechanics as follows. Let \mathbf{Q} be the linear map from the space $\mathcal{K}(\mathbb{C})$ to the space of unbounded operators acting on $L^2(\mathbb{R}, \mathbb{C})$ which is defined by

$$1 \mapsto Id, \quad x \mapsto x, \quad y \mapsto i\hbar \frac{\partial}{\partial x}, \quad \frac{x^2 + y^2}{2} \mapsto -\frac{\hbar^2}{2} \frac{\partial^2}{\partial x^2} + \frac{1}{2} x^2 - \left(\frac{\hbar^2}{8} + \frac{1}{2} \right). \quad (103)$$

Observe that \mathbf{Q} is “essentially” the operator which quantizes the classical harmonic oscillator.

Proposition 6.1. *For all $f \in \mathcal{K}(\mathbb{C})$ and for all $z \in \mathbb{C}$, we have :*

$$f(z) = \langle \Psi(z), \mathbf{Q}(f) \cdot \Psi(z) \rangle, \quad (104)$$

where $\langle \cdot, \cdot \rangle$ is the usual L^2 -scalar product on $L^2(\mathbb{R}, \mathbb{C})$.

Proof. By direct calculations. □

Because of the above proposition, we shall call $\Psi : \mathbb{C} \rightarrow L^2(\mathbb{R}, \mathbb{C})$ the *wave function* associated to $\mathcal{N}(\mu, 1)$. Clearly, this wave function comes from the embedding $\mathcal{N}(\mu, 1) \subseteq \mathcal{D}(\mathbb{R})$ together with the fact that every element of $T\mathcal{D}$ possesses a wave function (up to a phase, see §3).

More generally, if S is a submanifold of the space \mathcal{D} of probability density functions defined on an oriented (compact and connected) Riemannian manifold (M, g) , then $TS \subseteq T\mathcal{D}$, and thus, to every time-dependant probability density function ρ , there is, by solving the equation $\dot{\rho} = \operatorname{div}(\rho \nabla \phi)$, an associated wave function $\Psi = \sqrt{\rho} e^{-\frac{i}{\hbar} \phi}$. We thus get a map that we call the *wave function* associated to S (which is, strictly speaking, only defined up to a phase factor) :

$$\Psi : TS \rightarrow L^2(M, \mathbb{C}). \quad (105)$$

The above wave function is an infinite dimensional generalization of a wave function that we already considered¹¹ in [Mol]. In the latter paper, we consider a finite set $\Omega := \{x_1, \dots, x_n\}$ on which we define the

¹¹In [Mol], we use a different notation.

space \mathcal{P}_n^\times of positive probabilities p on Ω , i.e. $p : \Omega \rightarrow \mathbb{R}$, $p > 0$, $\sum_{k=1}^n p(x_k) = 1$. The space \mathcal{P}_n^\times is a finite dimensional statistical manifold. If $z_p = dp(t)/dt|_0$ is a tangent vector at $p \in \mathcal{P}_n^\times$, then we construct a wave function $\Psi : T\mathcal{P}_n^\times \rightarrow L^2(\Omega, \mathbb{C}) \cong \mathbb{C}^n$ as follows:

$$\Psi(z_p)(x_k) := \sqrt{p(x_k)} e^{iu_k/2}, \quad (106)$$

where $u_k \in \mathbb{R}$ is defined, for $k = 1, \dots, n$, by

$$\left. \frac{dp(t)(x_k)}{dt} \right|_0 = u_k p(x_k). \quad (107)$$

Equation (107) is a finite dimensional analogue of (99).

Using the above “finite dimensional” wave function, we were able in [Mol] to establish an analogue of Proposition 6.1 in the case of the binomial distribution $\mathcal{B}(n, q)$ defined over $\{0, 1, \dots, n\}$, the latter being viewed as a subspace of \mathcal{P}_{n+1}^\times (see [Mol], Proposition 9.7 and Lemma 9.8), and to conclude that the spin of particle in a Stern-Gerlach experiment is encoded in $\mathcal{B}(n, q)$.

These examples suggest that a “moving probability density function” always possesses an associated wave function, and that the latter, in good cases, is related to representation theory, quantization, and of course to the natural almost Hermitian structure of the underlying statistical manifold. More important, this suggests that the usual concepts of the standard quantum formalism (wave functions, Hilbert spaces, Hermitian operators, etc.) may be mathematically derived from more primitive concepts, rooted in statistics and information geometry.

To clarify these foundational aspects of quantum mechanics would be particularly interesting, especially in view of quantum gravity.

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References

- [AM78] R. Abraham and J. E. Marsden. *Foundations of mechanics*. Benjamin/Cummings Publishing Co., Reading, Mass., 1978. Second edition, revised and enlarged, With the assistance of Tudor Ratiu and Richard Cushman.
- [AN00] Shun-ichi Amari and Hiroshi Nagaoka. *Methods of information geometry*, volume 191 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 2000. Translated from the 1993 Japanese original by Daishi Harada.
- [AS99] Abhay Ashtekar and Troy A. Schilling. Geometrical formulation of quantum mechanics. In *On Einstein’s path (New York, 1996)*, pages 23–65. Springer, New York, 1999.
- [Dom62] Peter Dombrowski. On the geometry of the tangent bundle. *J. Reine Angew. Math.*, 210:73–88, 1962.
- [Ham82] R. S. Hamilton. *The inverse function theorem of Nash and Moser*. Bull. Amer. Math. Soc. (N.S.), 7(1):65–222, 1982.

- [Jos02] J. Jost. *Riemannian geometry and geometric analysis*. Universitext. Springer-Verlag, Berlin, third edition, 2002.
- [KM97] A. Kriegl and P. W. Michor. *The convenient setting of global analysis*, volume 53 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1997.
- [Lan02] S. Lang. *Introduction to differentiable manifolds*. Universitext. Springer-Verlag, New York, second edition, 2002.
- [Mol] M. Molitor. *Exponential families, Kähler geometry and quantum mechanics*. arxiv.org/abs/1203.2056v1.
- [Mol12] M. Molitor. Remarks on the statistical origin of the geometrical formulation of quantum mechanics. To appear in International Journal of Geometric Methods in Modern Physics, 9(3), 2012 (DOI: 10.1142/S0219887812200010).
- [Mol10] M. Molitor. *The group of unimodular automorphisms of a principal bundle and the Euler-Yang-Mills equations*. Differential Geometry and its Applications, 28(5):543–564, 2010.
- [MR93] Michael K. Murray and John W. Rice. *Differential geometry and statistics*, volume 48 of *Monographs on Statistics and Applied Probability*. Chapman & Hall, London, 1993.
- [Omo97] H. Omori. *Infinite-dimensional Lie groups*, volume 158 of Translations of Mathematical Monographs. American Mathematical Society, Providence, RI, 1997. Translated from the 1979 Japanese original and revised by the author.